MATH 3423 Homework 7 Due Fri, 11/1/13

Problem 1. Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u} := \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix} , \qquad \mathbf{v} := \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} , \qquad \mathbf{w} := \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} .$$

Directly from the definition, find out whether they are linearly dependent. Please write all your calculations in detail

Problem 2. Let V be a two-dimensional linear space, and $\mathsf{A} : V \to V$ be a linear operator acting on V. Let the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ form a basis in V, and the linear operator A acts on these two vectors as follows: $\mathsf{A}\mathbf{f}_1 = 2\mathbf{f}_1$, $\mathsf{A}\mathbf{f}_2 = 4\mathbf{f}_1 + \mathbf{f}_2$.

(a) Write down the matrix of the operator A in the basis $\mathbf{f}_1$, $\mathbf{f}_2$.

(b) If $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$, find $\mathsf{A}\mathbf{u}$ (expressed as a linear combination of the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$).

(c) Show that the vectors $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$ and $\mathbf{v} = 2\mathbf{f}_1 + \mathbf{f}_2$ are linearly independent, i.e., that $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$, (where $\alpha, \beta \in \mathbb{R}$), then $\alpha = 0$ and $\beta = 0$.

Problem 3. Let V be a linear space without any additional structure on it (i.e., there is no norm, inner product, or the concept of orthogonality). Consider the linear operator $\mathsf{A} : V \to V$ acting on V. Let $\mathbf{f}_1, \ldots, \mathbf{f}_n$ be a basis in V. One can define the matrix elements of A in this basis as follows:

$$\mathsf{A}\mathbf{f}_j =: \sum_{i=1}^{n} a_{ij}\, \mathbf{f}_i \ ,$$

so that the matrix of A in this basis is $\underline{\underline{A}} = (a_{ij})$ (see equation (7) on page 457).

(a) Let $\mathbf{u} = \sum_{j=1}^{n} u_j\, \mathbf{f}_j$. Find the components of $\mathsf{A}\mathbf{u}$ in the basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$; in other words, if $\mathbf{v} = \mathsf{A}\mathbf{u}$, find the numbers v_i such that $\mathbf{v} = \sum_{i=1}^{n} v_i\, \mathbf{f}_i$.

(b) Let B be another linear operator with matrix elements (b_{ij}) in the basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$. Find the matrix elements of the operator AB in the same basis. Here AB stands the composition of the operators in the sense that $(\mathsf{AB})\mathbf{u} := \mathsf{A}(\mathsf{B}\mathbf{u})$.

(c) Let $\mathsf{C} : V \to V$ be an *invertible* linear operator, i.e., a linear operator that has an inverse C^{-1}, so that $\mathsf{C}(\mathsf{C}^{-1}) = (\mathsf{C}^{-1})\mathsf{C} = \mathsf{I}$ (where I is the identity operator: $\mathsf{I}\mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$). Let C and C^{-1} have matrices $\underline{\underline{C}} = (c_{ij})$ and $\underline{\underline{D}} = (d_{ij})$, in the basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$:

$$\mathsf{C}\mathbf{f}_j = \sum_{i=1}^{n} c_{ij}\, \mathbf{f}_i \ , \qquad \mathsf{C}^{-1}\mathbf{f}_j = \sum_{i=1}^{n} d_{ij}\, \mathbf{f}_i \ .$$

Show that the matrix $\underline{\underline{D}}$ corresponding to C^{-1} is equal to the inverse of the matrix $\underline{\underline{C}}$ corresponding to C, i.e., $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$.

(d) Let us use the operators C and C^{-1} to define a new basis in V. Namely, let the new basis $\tilde{\mathbf{f}}_1, \ldots, \tilde{\mathbf{f}}_n$ be defined by

$$\mathbf{f}_j = \mathsf{C}\,\tilde{\mathbf{f}}_j = \sum_{i=1}^{n} c_{ij}\,\tilde{\mathbf{f}}_i \ ;$$

clearly, this implies that

$$\tilde{\mathbf{f}}_j = \mathsf{C}^{-1}\,\mathbf{f}_j = \sum_{i=1}^{n} d_{ij}\,\mathbf{f}_i \ .$$

Prove that if $\tilde{u}_j$ are the components of $\mathbf{u}$ in the new basis, i.e., $\mathbf{u} = \sum_{j=1}^{n} \tilde{u}_j \tilde{\mathbf{f}}_j$, then $\tilde{u}_i = \sum_{j=1}^{n} c_{ij} u_j$.

(e) Now we want to find the matrix of the operator A in the new basis $\tilde{\mathbf{f}}_1, \ldots, \tilde{\mathbf{f}}_n$. Define the matrix elements $\tilde{a}_{ij}$ of A in the new basis by

$$\mathsf{A}\tilde{\mathbf{f}}_j =: \sum_{i=1}^{n} \tilde{a}_{ij}\,\tilde{\mathbf{f}}_i \ ,$$

and let $\underline{\underline{\tilde{A}}} = (\tilde{a}_{ij})$. Show that $\underline{\underline{\tilde{A}}} = \underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}$.

Problem 4. Consider the linear space of polynomials of degree no greater than 3. Let us choose $\mathbf{f}_0 = 1$, $\mathbf{f}_1 = x$, $\mathbf{f}_2 = x^2$, $\mathbf{f}_3 = x^3$ to be a basis in this vector space, so that each polynomial $P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$ can be written as a vector $\mathbf{p}$ in this vector space as

$$\mathbf{p} = \sum_{i=0}^{3} p_i \mathbf{f}_i \ .$$

(a) Let D be the differentiation operator. Find the matrix $\underline{\underline{D}}$ of D in the basis $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$.

(b) Find the matrix of the operator D^2 (where $\mathsf{D}^2 := \mathsf{DD}$). You may use the fact that the matrix of a composition of the operators A and B is equal to the product of the matrices of A and B.

(c) Find the matrices of D^k for $k = 3, 4, \ldots$. Do you get the zero operator O for some value of k? An operator A for which it happens that $\mathsf{A}^k = \mathsf{O}$ for some k is called a *nilpotent* operator.

Problem 5. Let $V_n(a, b; w(x))$ stand for the linear space of polynomials of degree no greater than n endowed with the inner product

$$\langle P, Q\rangle = \int_a^b P(x)\, Q(x)\, w(x)\, \mathrm{d}x\ .$$

We want to construct polynomials D_0, D_1, ..., D_n satisfying the following conditions:

(i) the polynomial D_k is of degree k;

(ii) then coefficient of x^k in D_k is equal to 1 (such polynomials are called *monic*);

(iii) the polynomials D_0, D_1, D_2, ..., D_n form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^\infty x^k\, \mathrm{e}^{-x}\, \mathrm{d}x = k!$$

(where, by definition, $0! = 1$).

(a) Clearly, $D_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial D_1 of degree 1 that is orthogonal to D_0.

(b) Find the only monic quadratic polynomial D_2 that is orthogonal to both D_0 and D_1.

(c) Show that the polynomial $P(x) = x^2 + 3$ can be represented as a linear combination of the polynomials D_0, D_1 and D_2 as follows: $P = D_2 + 4D_1 + 5D_0$.

(d) Show by direct integration that $\langle D_0, D_0\rangle = 1$, $\langle D_1, D_1\rangle = 1$, $\langle D_2, D_2\rangle = 4$.

(e) Find the orthogonal projection, $\mathrm{proj}_{D_0+2D_1} Q$, of the polynomial $Q(x) = x^2 + 3$ onto the " straight line"

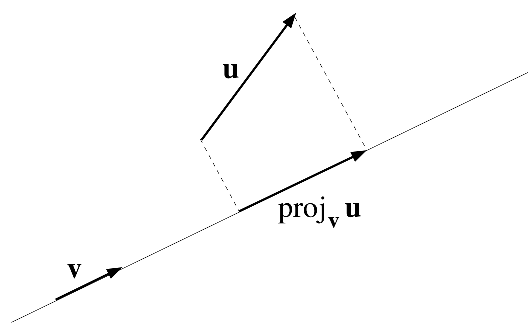
$$\ell := \{t(D_0 + 2D_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; \mathrm{e}^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space V, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\, \mathbf{v}$$

– see the picture below.



(f) Finally, let $\widetilde{D}_k := \mu_k D_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\left\| \widetilde{D}_k \right\| := \sqrt{\langle \widetilde{D}_k, \widetilde{D}_k \rangle}\ ,$$

of the polynomial $\widetilde{D}_k$ is 1. Find the explicit expressions for $\widetilde{D}_0(x)$, $\widetilde{D}_1(x)$, and $\widetilde{D}_2(x)$.

Problem 6. As we mentioned in class, one can define different norms in the same linear space. In this problem you will study different norms in $\mathbb{R}^2$. Let $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = u_1\mathbf{i} + u_2\mathbf{j} \in \mathbb{R}^2$.

(a) Define the norm $\|\mathbf{u}\|_2$ by

$$\|\mathbf{u}\|_2 := \sqrt{u_1^2 + u_2^2}\ .$$

Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \ : \ \|\mathbf{v}\|_2 \le 1\}$.

(b) Define the norm $\|\mathbf{u}\|_1$ by

$$\|\mathbf{u}\|_1 := |u_1| + |u_2|\ .$$

Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \ : \ \|\mathbf{v}\|_1 \le 1\}$.

(c) Define the norm $\|\mathbf{u}\|_\infty$ by

$$\|\mathbf{u}\|_\infty := \max\{|u_1|, |u_2|\}\ .$$

Draw the unit disk in $\mathbb{R}^2$ in this norm, i.e., the set of vectors $\{\mathbf{v} \in \mathbb{R}^2 \ : \ \|\mathbf{v}\|_\infty \le 1\}$.

(d) Two norms $\| \ \|$ and $\| \ \|'$ on the same linear space are said to be *equivalent* if there exist positive constants C_1 and C_2 such that $C_1 \|\mathbf{u}\| \le \|\mathbf{u}\|' \le C_2 \|\mathbf{u}\|$ for any vector $\mathbf{u} \in V$. In class we proved that $\|\mathbf{u}\|_\infty \le \|\mathbf{u}\|_2 \le \sqrt{2}\, \|\mathbf{u}\|_\infty$, i.e., that the norms $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent.

Show that the norms $\|\mathbf{u}\|_1$ and $\|\mathbf{u}\|_\infty$ are equivalent (you have to find the corresponding constants C_1 and C_2).

(e) Use the fact (proved in class) that the norms $\| \ \|_2$ and $\| \ \|_\infty$ are equivalent and the fact (proved in part (d)) that the norms $\| \ \|_1$ and $\| \ \|_\infty$ are equivalent to prove that the norms $\|\mathbf{u}\|_1$ and $\|\mathbf{u}\|_2$ are equivalent.