MATH 4093/5093 **Homework 7** **Due Fri, 4/8/2011**

Problem 1. In this problem you will show that some functions are Lipschitz and will find their Lipschitz constants.

(a) Let the function $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(t, y) = y \cos t$. Show that f is Lipschitz in y on $\mathbb{R}^2$, and the value of the Lipschitz constant is $L = 1$.

Hint: This example is practically the same as the first part of Example 7.3 on page 542 of the book.

(b) Let $D = \big\{(t, y) \in \mathbb{R}^2 \,:\, 0 \le t \le 3\,,\, 0 \le y \le \frac{\pi}{4}\big\}$, and let $f : D \to \mathbb{R} : (t, y) \mapsto \frac{t}{t+1} \tan y$. Show that f is Lipschits in y on D and find the value of the Lipschits constant.

Hint: Apply the same ideas as in Example 7.4 on page 544 of the book, in which the theorem on page 543 is used to compute the Lipschitz constant of a function.

Problem 2.

(a) Show that the function $y(t)$ defined implicitly by the equation

$$y(t)^2 - 2t\mathrm{e}^{-y(t)} - 4t^2 - 1 = 0 \tag{1}$$

satisfies the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{4t + \mathrm{e}^{-y}}{y + t\mathrm{e}^{-y}}\,, \qquad t \ge 0\,, \\ y(0) &= 1\,. \end{aligned} \tag{2}$$

(b) I found that

$$y(\tfrac{1}{2}) \approx 1.4916441946682422359111377449738198852929380037924846629297 1900605\,.$$

To find this value, I plugged $t = \frac{1}{2}$ in (1) to get $y(\frac{1}{2})^2 - \mathrm{e}^{-y(\frac{1}{2})} - 2 = 0$, and solved this equation numerically (in Mathematica) using Newton's method.

From the class web-site download the MATLAB code `euler.m`, which solves an IVP for a system of ODEs by Euler's method. Solve the IVP (2) and find $y(\frac{1}{2})$ numerically, using Euler's method. Do it with N = 10, 100, 1000, 10000, and 100000 (which corresponds to stepsize h = 0.05, 0.005, 0.0005, 0.00005 and 0.000005, respectively). In a table, put the values of N, the corresponding values of $y(\frac{1}{2})_{\text{approx}}$ obtained by running `euler.m`, as well as the absolute errors $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, where $y(\frac{1}{2})_{\text{exact}}$ is the exact value. Use small enough tolerance, i.e., 10^{-12}. Please attach a printout of your MATLAB session (there is no need to use MATLAB to print the table with the data, just copy the data needed from your MATLAB printout in your table by hand).

Remark: Note that the code `euler.m` can solve an IVP for a system of ODEs; it finds the number of equations in your system from the number of initial conditions you give it. In this problem, the initial condition you will give to the MATLAB program is only one, so the code will understand that you are solving an IVP for a single ODE.

(c) In MATLAB, plot the logarithm of the error, $|y(\frac{1}{2})_{\text{exact}} - y(\frac{1}{2})_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the approximate straight line that goes through these points. How does the value of this slope match with the theoretical prediction for the value of the error of Euler's method?

Remarks: 1) Note that you can use natural logarithms or logarithms base 10, or any other base to plot the results (but use the same base for both axes!) – this is not going to change the slope of the approximate straight line.

2) To plot, say, the points (x_j, y_j) (for $j = 1, 2, 3, 4, 5$), where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) = (4, 5, 6, 7, 8)$ and $\mathbf{y} = (y_1, y_2, y_3, y_4, y_5) = (12, 11, 12, 10, 9)$, you can type the following MATLAB commands:

```
x = [4 5 6 7 8]
y = [12 11 12 10 9]
plot(x,y)
```

If, instead of `plot(x,y)` you type `plot(x,y,'-*')`, the data will be represented by stars connected by straight lines. Type `help plot` for more information.

Problem 3. Consider the IVP

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= y^2 + \frac{1}{t^2}\,, \qquad t \in [1, 2]\,, \\ y(1) &= -\frac{1}{2}\,. \end{aligned} \tag{3}$$

(a) Develop Taylor's method of order 2 to solve the IVP (3). Please write your derivations in detail.

Hint: In other words, find the derivative of the right-hand side of the ODE in (3) (which represents the second derivative $y''(t)$ of the exact solution), and write expressions for the values of w_0 and w_{i+1} (where w_{i+1} should be expressed in terms of the values of w_i, t_i and the stepsize h.

(b) Use the MATLAB code `taylor2nd.m` from the class web-site and a MATLAB file that will supply the right-hand side of the ODE and the first derivative of the right-hand side of the ODE with respect to t, to solve the IVP (3) by using Taylor method of order 2 in order to find the value of $y(2)$, with $N = 10$, 100, 1000, and 10000. Record the numerical values of $y(2)$ obtained by using different N. Attach your MATLAB printout.

Hint: In class we considered the ODE $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 + \frac{y}{t}$, and found that $\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + \frac{y(t)}{t}\right) = \frac{1}{t}$. If I want to create a MATLAB function `fun1_and_fun1der` that takes the values of t and y and returns the right-hand side and its first derivative, then I can create a MATLAB file `fun1_and_fun1der.m` that looks like this:

```
function [fun1, fun1der] = fun1_and_fun1der(t,y)
     fun1 = 1 + y/t;
     fun1der = 1/t;
```

(c) Plot in MATLAB the logarithm of the error, $|y(2)_{\text{exact}} - y(2)_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP (3) is

$$y(t)_{\text{exact}} = \tfrac{1}{2t}\left[\sqrt{3}\tan\left(\tfrac{\sqrt{3}}{2}\ln|t|\right) - 1\right] . \tag{4}$$

Problem 4. Consider the same IVP as in Problem 3.

(a) Develop Taylor's method of order 3 to solve the IVP (3).

Hint: Here is what I obtained for the second derivative of $f(t, y(t))$:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} f(t, y(t)) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(y(t)^2 + \frac{1}{t^2}\right) = 6y^4(t) + \frac{8y^2(t)}{t^2} - \frac{4y(t)}{t^3} + \frac{8}{t^4} .$$

(b) Write a MATLAB code `yourfirstname_yourfamilyname_taylor3rd.m` that solves IVPs for ODEs by using Taylor's method of order 3 Perhaps the easiest way to do this will be to take `taylor2nd.m` and to modify it (the modification will be really minor). Do not forget to the change first line of the file to

`function [wi,ti] = yourfirstname_yourfamilyname_taylor3rd(RHS,t0,x0,tf,N)`

– otherwise MATLAB will complain. Write a MATLAB file that returns the right-hand side of the ODE and all the derivatives that are needed by `taylor3rd.m`. Put your code `yourfirstname_yourfamilyname_taylor3rd.m` in the Dropbox of D2L (`learn.ou.edu`), and attach a printout of your MATLAB codes to your homework.

(c) Run `yourfirstname_yourfamilyname_taylor3rd.m` to solve the IVP (3) with $N =$ 10, 100, 1000, and 10000. Record the numerical values of $y(2)$ obtained by using different N in a table.

(d) Plot in MATLAB the logarithm of the error, $|y(2)_{\text{exact}} - y(2)_{\text{approx}}|$, versus the logarithm of the stepsize h. Find the slope of the straight line through the points on your graph, and discuss how this value compares with the theoretical prediction. The exact solution of the IVP is given in (4).