MATH 4443 **Homework 7** **Due 3/15/18 (Thursday)**

Problem 1. Let

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Show that f is integrable on $[0, 1]$ and compute $\int_0^1 f$.

Hint: Consider the set of points

$$D_{\varepsilon/2} = \left\{ x \in [0, 1] : f(x) \geq \tfrac{\varepsilon}{2} \right\} \ .$$

Take a partition P such that the intervals containing points from $D_{\varepsilon/2}$ have a total length $\frac{\varepsilon}{2}$. It will then be easy to give an upper bound on $U(f, P)$.

Problem 2. Provide an example of the following; explain your reasoning, draw a picture if needed.

(a) A sequence (f_n) that converges to f pointwise, where each f_n has at most a finite number of discontinuities (hence is integrable), but f is not integrable.

(b) A sequence (g_n) that converges uniformly to g, where each g_n is not integrable, but g is integrable.

(c) A non-integrable function h such that $|h|$ is integrable.

(d) A function $r : [a, b] \to \mathbb{R}$ that is non-negative (i.e., $r(x) \geq 0$ for all $x \in [a, b]$) and such that $r(x) > 0$ for an infinite number of points $x \in [a, b]$, but $\int_a^b r = 0$.

(e) A sequence (t_n) that converges to 0 pointwise such that each function t_n is integrable on $[a, b]$ but $\displaystyle\lim_{n\to\infty} \int_a^b t_n$ does not exist.

(f) A sequence (u_n) of non-negative functions u_n with $\displaystyle\lim_{n\to\infty} \int_0^1 u_n = 0$, but such that $u_n(x)$ does not converge to 0 for any $x \in [0, 1]$.

Problem 3. Prove that, if f is continuous on $[a, b]$ and $f(x) \geq 0$ for all $x \in [a, b]$ with $f(x_0) > 0$ for at least one point $x_0 \in [a, b]$, then $\int_a^b f > 0$.

Problem 4. Although this was not a part of Theorem 7.4.2, it is true that the product of integrable functions is integrable. Provide the details for each step in the following proof of this fact; you may use Theorem 7.4.2, but please indicate clearly which part of that theorem you are using.

(a) If f satisfies $|f(x)| \leq M$ on $[a, b]$, show that

$$\left|(f(x))^2 - (f(y))^2\right| \leq 2M|f(x) - f(y)| \ .$$

(b) Prove that, if f is integrable on $[a, b]$, then so is f^2.

(c) Prove that, if f and g are integrable, then fg is integrable.

Hint: Consider $(f + g)^2$.

Problem 5. In this problem you will give a detailed proof of Theorem 7.4.2(v), i.e., you will show that that the integrability of f implies the integrability of $|f|$, and will obtain a useful inequality between the integrals of these functions. Let $f : A \to \mathbb{R}$ be a bounded function, and set

$$M = \sup\{f(x) : x \in A\} \ , \qquad m = \inf\{f(x) : x \in A\} \ ,$$
$$M' = \sup\{|f(x)| : x \in A\} \ , \qquad m' = \inf\{|f(x)| : x \in A\} \ , \qquad .$$

(a) Prove that $M - m \geq M' - m'$.

Hint: You may need the inequality $||a| - |b|| \leq |a - b|$ (which follows easily from the triangle inequality), and the useful characterization of sup given in Lemma 1.3.8 (and a similar characterization of inf).

(b) Show that, if f is integrable on the interval $[a, b]$, then $|f|$ is also integrable on $[a, b]$.

(c) Prove that, if f is integrable on $[a, b]$, then $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Food for Thought: Aksoy & Khamsi, Problems 7.8, 7.9(a), 7.15 (all solved in the book).