MATH 4093/5093 Homework 7 Due Fri, 11/12/10

Problem 1. Let
$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} .$$

(a) Prove the the pair of matrices
$$L_1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} , \qquad U_1 = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$$
forms an LU decomposition of A. Also show that the pair of matrices
$$L_2 = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} , \qquad U_2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
forms an LU decomposition of A.

(b) In class we discussed that the arbitrariness in the choice of matrices L and U in $A = LU$ is in the choice of a non-singular diagonal matrix D such that $L_1 = L_2 D$ and $U_1 = D^{-1} U_2$. Find explicitly the matrix D for the pairs (L_1, U_1) and (L_2, U_2) given in part (a).

(c) Use the pair (L_1, U_1) from part (a) to solve the system $A\mathbf{x} = (\ 4 \ \ 6\ )^T$.

(d) Let
$$B = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} .$$
Construct a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U, such that
$$PB = LU .$$
Hint: You can do this with almost no additional calculations if you look carefully at the matrices A and B.

Problem 2. Let
$$A = \begin{pmatrix} 2 & 7 & 5 \\ 6 & 20 & 10 \\ 4 & 3 & 0 \end{pmatrix} .$$

(a) Show by direct computation that
$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 + \alpha a_1 & b_2 + \alpha a_2 & b_3 + \alpha a_3 \\ c_1 + \beta a_1 & c_2 + \beta a_2 & c_3 + \beta a_3 \end{pmatrix} .$$

Use this fact to find a matrix

$$M_1 = \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}$$

so that

$$M_1 A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} ,$$

where the stars represent arbitrary numbers.

(b) Show by direct computation that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma & 1 \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 + \gamma b_1 & c_2 + \gamma b_2 & c_3 + \gamma b_3 \end{pmatrix} ,$$

Use this fact to find a matrix

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma & 1 \end{pmatrix}$$

so that

$$M_2 M_1 A = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

(again, the stars stand for arbitrary numbers).

(c) Show that

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ \beta & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\alpha & 1 & 0 \\ -\beta & 0 & 1 \end{pmatrix} , \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \gamma & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\gamma & 1 \end{pmatrix} .$$

(d) Use your results from the previous parts of this problem to construct an explicit LU decomposition of the matrix A.

Problem 3. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} .$$

(a) Construct explicitly the matrices T_{J} and T_{GS} corresponding to the Jacobi and the Gauss-Seidel iterative methods.

(b) Determine the spectral radii of the iteration matrices $T_{\rm J}$ and $T_{\rm GS}$.

(c) Will the Jacobi method converge for any choice of initial vector $\mathbf{x}^{(0)}$? Will the Gauss-Seidel method converge for any choice of initial vector $\mathbf{x}^{(0)}$? Explain.

Problem 4. Consider the function $f(x) = \sqrt{x}$. We want to find a cubic polynomial $S(x)$ that interpolates the function $f(x)$ on the interval $x \in [1, 4]$ with clamped boundary conditions at both ends.

(a) Clearly, $S(x)$ must have the same values as $f(x)$ at the points $x_0 = 1$ and $x_1 = 4$. What are the clamped boundary conditions for $S'(1)$ and $S'(4)$?

(b) Write the interpolating polynomial $S(x)$ in the form

$$S(x) = a + b(x-1) + c(x-1)^2 + d(x-1)^3 \ ,$$

and impose the conditions formulated in part (a) to find the coefficients of $S(x)$. You should obtain that $c = -\frac{1}{12}$ and $d = \frac{1}{108}$.

(c) Apply the Theorem on page 402 to give a rigorous upper bound on the interpolation error. Compare this numerical value with the actual value of the error, which is about 0.013.

(d) The function $S(x)$ that you found in (a) is good not only to approximate the values of the function $f(x)$, but also to approximate the values of the integrals and some derivatives of $f(x)$. Find the numerical value of $\int_1^3 S(x)\,dx$ and compare it with the exact value, $\int_1^3 f(x)\,dx$; find the absolute and the relative errors.