MATH 5163 **Homework 7** **Due Fri, 3/6/14, 5 p.m.**

Problem 1. [Green function for Dirichlet BVP for Poisson equation in half-ball] In this problem you will use the methods of electrostatic images to construct the Green function for the Dirichlet boundary value problem for Poisson equation in a half-ball of radius R in $\mathbb{R}^3$, i.e., a function $G(\mathbf{x},\mathbf{y})$ satisfying

$$\begin{aligned} -\Delta_{\mathbf{y}} G(\mathbf{x},\mathbf{y}) &= \delta_{\mathbf{x}}(\mathbf{y}) , \qquad && \text{for } \mathbf{y}\in\Omega , \\ G(\mathbf{x},\mathbf{y}) &= 0 , && \text{for } \mathbf{y}\in\partial\Omega , \end{aligned}$$

where $\Omega := \{\mathbf{x}\in\mathbb{R}^3 : |\mathbf{x}| < R,\ x_3 > 0\}$ is the half-ball of radius R in $\mathbb{R}^3$, and $\mathbf{x} = (x_1,x_2,x_3)$ is an arbitrary point in Ω (i.e., $x_1^2 + x_2^2 < R^2$ and $x_3 > 0$). Please follow the steps below.

(a) Let $\mathbf{y}$ be a point on the flat part of $\partial\Omega$, i.e., $\mathbf{y} = (y_1,y_2,0)$ with $y_1^2 + y_2^2 < R^2$. To make sure that $G(\mathbf{x},\mathbf{y}) = 0$, you have to place an image charge at the point $\mathbf{x}^* = (x_1,x_2,-x_3)$. Draw a picture indicating the position of $\mathbf{x}$ and $\mathbf{x}^*$.

(b) Now ignore the image charge at $\mathbf{x}^*$. If we only had the charge at the point $\mathbf{x}$, at what position $\overline{\mathbf{x}}$ would you place an image charge so that the potential of both charges is zero on the sphere of radius R? In other words, if at the point $\mathbf{y}$ the potential due to the charges at $\mathbf{x}$ and at $\overline{\mathbf{x}}$ is $G(\mathbf{x},\mathbf{y})$, then $G(\mathbf{x},\mathbf{y})$ must be zero when $\mathbf{y}\in\partial B_R(\mathbf{0})$. Draw a picture indicating the position of of $\mathbf{x}$ and $\overline{\mathbf{x}}$.

(c) Draw a picture with the charges at $\mathbf{x}$, $\mathbf{x}^*$, and $\overline{\mathbf{x}}$. If both image charges (from part (a) and from part (b)) are present, then $G(\mathbf{x},\mathbf{y})$ does not vanish when $\mathbf{y}\in\partial\Omega$ (why?). However, you can place only one more image charge at certain point – let's denote it by $\overline{\mathbf{x}}^*$ – so that if the value $G(\mathbf{x},\mathbf{y})$ at $\mathbf{y}\in\overline{\Omega}$ of the potential due to the "true" charge at $\mathbf{x}\in\Omega$ and the three image charges at $\mathbf{x}^*\notin\Omega$, $\overline{\mathbf{x}}\notin\Omega$, and $\overline{\mathbf{x}}^*\notin\Omega$, then $G(\mathbf{x},\mathbf{y}) = 0$ when $\mathbf{y}\in\partial\Omega$. Please draw a picture with the locations of $\mathbf{x}$, $\mathbf{x}^*$, $\overline{\mathbf{x}}$, and $\overline{\mathbf{x}}^*$, write explicitly $\overline{\mathbf{x}}^*$ and the image charge you have to place at this point.

(d) Write down the Green function $G(\mathbf{x},\mathbf{y})$ constructed above.

(e) Write down the corresponding Poisson kernel,

$$P(\mathbf{x},\mathbf{y}) = -\frac{\partial G}{\partial \nu_{\mathbf{y}}}(\mathbf{x},\mathbf{y})\Big|_{\mathbf{y}\in\partial\Omega}$$

when $\mathbf{y}$ belongs to the flat part of $\partial\Omega$. (Of course, one can also write $P(\mathbf{x},\mathbf{y})$ for $\mathbf{y}$ in the curved part of $\partial\Omega$, but the calculations are too tedious.)

(f) If $G(\mathbf{x},\mathbf{y})$ and $P(\mathbf{x},\mathbf{y})$ are the Green function and the Poisson kernel obtained above, write down the solution $u(\mathbf{x})$ of the boundary value problem (just write down the abstract expressions)

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}) , \qquad && \text{for } \mathbf{x}\in\Omega , \\ u(\mathbf{x}) &= g(\mathbf{x}) , && \text{for } \mathbf{x}\in\partial\Omega . \end{aligned}$$

Problem 2. [Poisson formula in the disk (2-dimensional ball) of radius R]
In Problem 5 of Homework 5 you proved that the function

$$\Phi(\mathbf{x}) = -\frac{1}{2\pi}\ln|\mathbf{x}| , \qquad \mathbf{x} \in \mathbb{R}^2 \tag{1}$$

is a fundamental solution of $-\Delta$ in $\mathbb{R}^2$, i.e.,

$$-\Delta\Phi(\mathbf{x}) = \delta(\mathbf{x}) , \qquad \mathbf{x} \in \mathbb{R}^2 .$$

In this problem you will use this fact and the method of electrostatic images to show that the Green function of the Dirichlet boundary value problem for the Poisson equation in a disk $B_R(\mathbf{0})$ of radius R centered in the origin of $\mathbb{R}^2$ is equal to

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi\left(\frac{|\mathbf{x}|}{R}(\mathbf{x}^*-\mathbf{y})\right) = \frac{1}{2\pi}\ln\frac{|\mathbf{x}|\,|\mathbf{x}^*-\mathbf{y}|}{R|\mathbf{x}-\mathbf{y}|} , \tag{2}$$

where $\mathbf{x}^* = \dfrac{R^2}{|\mathbf{x}|^2}\mathbf{x}$ is the image of $\mathbf{x}$ under inversion (note that $\mathbf{x}^* \notin B_R(\mathbf{0})$), and $\Phi(\mathbf{x})$ is the fundamental solution of the operator $-\Delta$ given by (1).

(a) Explain why, for an arbitrary $\mathbf{x} \in B_R(\mathbf{0})$, the function $G(\mathbf{x},\mathbf{y})$ given by (2) satisfies

$$-\Delta_{\mathbf{y}}G(\mathbf{x},\mathbf{y}) = \delta_{\mathbf{x}}(\mathbf{y}) , \qquad \mathbf{y} \in B_R(\mathbf{y}) .$$

(b) Check by a direct calculation that, if $\mathbf{x}$ is an arbitrary point in $B_R(\mathbf{0})$, $\mathbf{x}^*$ is its image under inversion, and $\mathbf{y} \in \partial B_R(\mathbf{0})$, then

$$\frac{|\mathbf{x}-\mathbf{y}|^2}{|\mathbf{x}^*-\mathbf{y}|^2} = \frac{|\mathbf{x}|^2}{R^2} ,$$

and explain why this implies that $G(\mathbf{x},\mathbf{y})$ is zero for $\mathbf{y} \in \partial B_R(\mathbf{0})$.

(c) Let $\mathbf{x}$ and $\mathbf{y}$ have polar coordinates (r,θ) and (ρ,α), respectively, i.e.,

$$\mathbf{x} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} , \qquad \mathbf{y} = \rho\cos\alpha\,\mathbf{i} + \rho\sin\alpha\,\mathbf{j}$$

(where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in positive x- and y-directions, respectively). Show that the expression (2) for $G(\mathbf{x},\mathbf{y})$ can be written as

$$G(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi}\ln\frac{R^4 + r^2\rho^2 - 2R^2 r\rho\cos(\theta-\alpha)}{R^2\left[r^2+\rho^2-2r\rho\cos(\theta-\alpha)\right]} .$$

(d) Show that the Poisson kernel (cf. Salsa's book, page 137) is

$$P(\mathbf{x},\mathbf{y}) := -\frac{\partial G}{\partial\nu_{\mathbf{y}}}(\mathbf{x},\mathbf{y})\bigg|_{\mathbf{y}\in\partial B_R(\mathbf{0})} = -\frac{\partial G}{\partial\rho}(\mathbf{x},\mathbf{y})\bigg|_{\rho=R} = \frac{1}{2\pi R}\,\frac{1-\left(\frac{r}{R}\right)^2}{\left(\frac{r}{R}\right)^2 + 1 - 2\frac{r}{R}\cos(\theta-\alpha)} ,$$

where $\mathbf{x} \in B_R(\mathbf{0})$, $\mathbf{y} \in \partial B_R(\mathbf{0})$.

(e) Use the expressions derived above to write down the solution of the boundary value problem

$$\begin{aligned} \Delta u(\mathbf{x}) &= 0 \ , \qquad && \text{for } \mathbf{x} \in B_R(\mathbf{0}) \ , \\ u(\mathbf{x}) &= g(\theta) \ , \qquad && \text{for } \mathbf{x} \in \partial B_R(\mathbf{0}) \ , \end{aligned}$$

where (r, θ) are the polar coordinates of the point $\mathbf{x}$.

Problem 3. [Translation operator on $\mathscr{D}'(\mathbb{R}^n)$]

Let $\mathbf{v} \in \mathbb{R}^n$, and define the operator $E_{\mathbf{v}}$ on $\mathscr{D}(\mathbb{R}^n)$ by $(E_{\mathbf{v}}\phi)(\mathbf{x}) := \phi(\mathbf{x} - \mathbf{v})$. It is easy to show that $E_{\mathbf{v}}$ is bijective (one-to-one and onto) from $\mathscr{D}(\mathbb{R}^n)$ to $\mathscr{D}(\mathbb{R}^n)$. Define the operator $E_{\mathbf{v}}$ on $\mathscr{D}'(\mathbb{R}^n)$ as follows: $\langle E_{\mathbf{v}}F, \phi\rangle := \langle F, \circ E_{-\mathbf{v}}\phi\rangle$ for any $\phi \in \mathscr{D}(\mathbb{R}^n)$. We give such a definition (with $-\mathbf{v}$) so that if F is a regular distribution, i.e., related to some $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ by $\langle F, \phi\rangle = \int_{\mathbb{R}} f(\mathbf{x})\,\phi(\mathbf{x})\,\mathrm{d}x$, then $E_{\mathbf{v}}F$ will correspond to the function $E_{\mathbf{v}}f$: indeed,

$$\begin{aligned} \langle E_{\mathbf{v}}F, \phi\rangle = \langle F, E_{-\mathbf{v}}\phi\rangle &= \int_{\mathbb{R}} f(\mathbf{x})\,(E_{-\mathbf{v}}\phi)(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}} f(\mathbf{x})\,\phi(\mathbf{x} + \mathbf{v})\,\mathrm{d}\mathbf{x} \\ &= \int_{\mathbb{R}} f(\mathbf{y} - \mathbf{v})\,\phi(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_{\mathbb{R}} (E_{\mathbf{v}}f)(\mathbf{y})\,\phi(\mathbf{y})\,\mathrm{d}\mathbf{y} \ . \end{aligned}$$

(a) Show that the mapping $E_{\mathbf{v}} : \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ is linear.

(b) Show that $E_{\mathbf{v}} : \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ is a continuous mapping.

(c) Show that the mapping $E_{\mathbf{v}} : \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ is injective, i.e., that if $E_{\mathbf{v}}F = E_{\mathbf{v}}G$ for $F, G \in \mathscr{D}'(\mathbb{R}^n)$, then $F = G$.

(d) Show that the mapping $E_{\mathbf{v}} : \mathscr{D}'(\mathbb{R}^n) \to \mathscr{D}'(\mathbb{R}^n)$ is surjective, i.e., that for every $F \in \mathscr{D}'(\mathbb{R}^n)$ there is $G \in \mathscr{D}'(\mathbb{R}^n)$ such that $E_{\mathbf{v}}G = G$.

Problem 4. [Fundamental solution in $\mathscr{D}'(\mathbb{R}^n)$]

A *fundamental solution* of a differential operator $A = \sum_{|\boldsymbol{\alpha}| \le m} c_{\boldsymbol{\alpha}} D^{\boldsymbol{\alpha}}$ (where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$ is a multiindex, and $|\boldsymbol{\alpha}| = \alpha_1 + \dots + \alpha_n$) is a distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ such that $AT = \sum_{|\boldsymbol{\alpha}| \le m} c_{\boldsymbol{\alpha}} D^{\boldsymbol{\alpha}} T = \delta$.

(a) Use what you know about convolution to show that if T is a fundamental solution of the operator A in $\mathscr{D}'(\mathbb{R}^n)$, then $A(T * \phi) = \phi$ for all $\phi \in \mathscr{D}(\mathbb{R}^n)$.

(b) Find a fundamental solution of the operator $\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}$ in $\mathscr{D}'(\mathbb{R})$. Look for it in the form

$$\langle T, \phi\rangle = \int_{\mathbb{R}} f(x)\,H(x)\,\phi(x)\,\mathrm{d}x = \int_0^\infty f(x)\,\phi(x)\,\mathrm{d}x \ , \qquad \phi \in \mathscr{D}(\mathbb{R}) \ ,$$

where $f \in C^2(\mathbb{R})$ and H is the Heaviside function, and show that f is the solution of the initial value problem $f''(x) = 0$ for $x \in (0, \infty)$, $f(0) = 0$, $f'(0) = 1$, whose solution is easy to find.