MATH 5463 Homework 7 Due Thu, Apr 9

Problem 8.1/2(a) from Folland's book.

Problem 1.

Let $p \in (1, \infty)$ and $f \in L^p((0,\infty), m)$ (where m is the Lebesgue measure on $(0,\infty)$). Define the map $L : f \mapsto Lf$ by

$$(Lf)(x) = \frac{1}{x} \int_0^x f(t)\,\mathrm{d}t \ .$$

(a) Prove Hardy's ineqality,

$$\|Lf\|_p \leq \frac{p}{p-1}\,\|f\|_p \ ,$$

which shows that L maps $L^p((0,\infty), m)$ into itself.

Hint: First assume that $f \geq 0$ and $f \in C_c((0,\infty))$. Set $F(x) = (Lf)(x)$ and integrate by parts to show that

$$\int_0^\infty F(x)^p\,\mathrm{d}x = -p \int_0^\infty F(x)^{p-1}\,xF'(x)\,\mathrm{d}x \ .$$

Show that $xF'(x) = f(x) - F(x)$ and use Hölder's inequality for $\int_0^\infty F(x)^{p-1} f(x)\,\mathrm{d}x$. Then derive the general case.

(b) Show that the constant $\frac{p}{p-1}$ cannot be replaced with a smaller one.

Hint: Set $f(x) = x^{-1/p}\,\chi_{[1,A]}(x)$ for some large A.

(c) Prove that if $f > 0$ and $f \in L^1((0,\infty), m)$, then $Lf \notin L^1((0,\infty), m)$.

Hint: One way of proving this is to change the order of integration in the double integral, similarly to what you did in Problem 1 of Homework 6.

Problem 2.

(a) Prove that if f is a measurable function on $\mathbb{R}$ and f is continuous at 0, then

$$\lim_{n\to\infty} n \int_{[0,\frac{1}{n}]} f(x)\,\mathrm{d}x = f(0) \ .$$

(b) Let f be a continuous function on $\mathbb{R}$. Show that

$$\lim_{n\to\infty} \int_{[0,1]} \left| f\left(x + \frac{1}{n}\right) - f(x) \right| \mathrm{d}x = 0 \ .$$

Problem 3.

(a) Suppose that f is monotone on $(0,1)$ and the improper Riemann integral $\int_0^1 f(x)\,\mathrm{d}x$ exists. Prove that

$$\lim_{n\to\infty} \frac{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n-1}{n}\right)}{n} = \int_0^1 f(x)\,\mathrm{d}x\ .$$

Hint: If f is monotonically increasing, use elementary Calculus to show that

$$\int_0^{1-\frac{1}{n}} f(x)\,\mathrm{d}x \le \frac{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{n-1}{n}\right)}{n} \le \int_{\frac{1}{n}}^1 f(x)\,\mathrm{d}x\ ,$$

and use this to derive the desired result. What about the case of a monotonically decreasing f?

(b) Now we generalize the problem from (a) to the case of Borel measures. Let μ_n be a sequence of finite Borel measures on $[0,1]$.

(b_1) Suppose that $\lim\limits_{n\to\infty} \int_{[0,1]} f(x)\,\mathrm{d}\mu_n(x)$ exists for every $f \in C([0,1])$. Show that there exists another finite Borel measure μ on $[0,1]$ such that

$$\lim_{n\to\infty} \int_{[0,1]} f(x)\,\mathrm{d}\mu_n(x) = \int_{[0,1]} f(x)\,\mathrm{d}\mu(x) \qquad \text{for all } f \in C([0,1])\ .$$

Hint: You only have to use one of the powerful theorems proved recently in class.

(b_2) Show that the sequence of measures $\mu_n = \dfrac{1}{n}\displaystyle\sum_{j=1}^n \delta_{j/n}$ satisfy the assumptions from part (b_1) (here δ_x stands for the Dirac measure at x). Identify the limit measure μ in this case.

Problem 4.

Suppose that $\{f_n\}$ is a sequence of measurable functions on $[0,1]$ with $\displaystyle\int_0^1 |f_n(x)|^2\,\mathrm{d}x \le 10$, and $f_n \to 0$ m-a.e. on $[0,1]$. Prove that $\displaystyle\int_0^1 |f_n(x)|\,\mathrm{d}x \to 0$.

Hint: Use Egoroff's Theorem and Hölder's inequality with $p = \frac{1}{2}$.