

Problem 8.1/2(a) from Folland's book.

Problem 1.

Let $p \in (1, \infty)$ and $f \in L^p((0, \infty), m)$ (where m is the Lebesgue measure on $(0, \infty)$). Define the map $L : f \mapsto Lf$ by

$$(Lf)(x) = \frac{1}{x} \int_0^x f(t) dt .$$

(a) Prove Hardy's inequality,

$$\|Lf\|_p \leq \frac{p}{p-1} \|f\|_p ,$$

which shows that L maps $L^p((0, \infty), m)$ into itself.

Hint: First assume that $f \geq 0$ and $f \in C_c((0, \infty))$. Set $F(x) = (Lf)(x)$ and integrate by parts to show that

$$\int_0^\infty F(x)^p dx = -p \int_0^\infty F(x)^{p-1} x F'(x) dx .$$

Show that $x F'(x) = f(x) - F(x)$ and use Hölder's inequality for $\int_0^\infty F(x)^{p-1} f(x) dx$. Then derive the general case.

(b) Show that the constant $\frac{p}{p-1}$ cannot be replaced with a smaller one.

Hint: Set $f(x) = x^{-1/p} \chi_{[1,A]}(x)$ for some large A .

(c) Prove that if $f > 0$ and $f \in L^1((0, \infty), m)$, then $Lf \notin L^1((0, \infty), m)$.

Hint: One way of proving this is to change the order of integration in the double integral, similarly to what you did in Problem 1 of Homework 6.

Problem 2.

(a) Prove that if f is a measurable function on \mathbb{R} and f is continuous at 0, then

$$\lim_{n \rightarrow \infty} n \int_{[0, \frac{1}{n}]} f(x) dx = f(0) .$$

(b) Let f be a continuous function on \mathbb{R} . Show that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} \left| f\left(x + \frac{1}{n}\right) - f(x) \right| dx = 0 .$$

Problem 3.

- (a) Suppose that f is monotone on $(0, 1)$ and the improper Riemann integral $\int_0^1 f(x) dx$ exists. Prove that

$$\lim_{n \rightarrow \infty} \frac{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)}{n} = \int_0^1 f(x) dx .$$

Hint: If f is monotonically increasing, use elementary Calculus to show that

$$\int_0^{1-\frac{1}{n}} f(x) dx \leq \frac{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)}{n} \leq \int_{\frac{1}{n}}^1 f(x) dx ,$$

and use this to derive the desired result. What about the case of a monotonically decreasing f ?

- (b) Now we generalize the problem from (a) to the case of Borel measures. Let μ_n be a sequence of finite Borel measures on $[0, 1]$.

- (b₁) Suppose that $\lim_{n \rightarrow \infty} \int_{[0,1]} f(x) d\mu_n(x)$ exists for every $f \in C([0, 1])$. Show that there exists another finite Borel measure μ on $[0, 1]$ such that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f(x) d\mu_n(x) = \int_{[0,1]} f(x) d\mu(x) \quad \text{for all } f \in C([0, 1]) .$$

Hint: You only have to use one of the powerful theorems proved recently in class.

- (b₂) Show that the sequence of measures $\mu_n = \frac{1}{n} \sum_{j=1}^n \delta_{j/n}$ satisfy the assumptions from part (b₁) (here δ_x stands for the Dirac measure at x). Identify the limit measure μ in this case.

Problem 4.

Suppose that $\{f_n\}$ is a sequence of measurable functions on $[0, 1]$ with $\int_0^1 |f_n(x)|^2 dx \leq 10$, and $f_n \rightarrow 0$ m -a.e. on $[0, 1]$. Prove that $\int_0^1 |f_n(x)| dx \rightarrow 0$.

Hint: Use Egoroff's Theorem and Hölder's inequality with $p = \frac{1}{2}$.