MATH 2513 Homework 8 Due 3/28/19 (Thursday)

Please explain briefly but clearly your reasoning (unless it is totally obvious from your answer, i.e., when you have to list the elements of a set or to draw a set in the plane).

Please write the problems in the same order as they are given in the assignment.

Note that the odd-numbered problems have answers at the end of Hammack's book. I strongly suggest that you do *all* odd-numbered problems for practice; moreover, many of them are very similar to the assigned homework problems.

Hammack, Chapter 6:

- Exercise 6 (hint: see p. 137 of the book),
- Exercise 18 (hint: to prove the statement by contradiction, assume that $4|(a^2+b^2)$ and a and b are both odd, then write $a = 2k+1$ and $b = 2n+1$ for some integers k and n, and work on $a^2 + b^2$; see also the hint to Additional Problem 3 below),
- Exercise 24 (hint: assume that $\log_2 3 = \frac{k}{n}$ for some integers k and n, with $n \neq 0$, and then use the definition of logarithm; we did something very similar in class on March 14).

Hammack, Chapter 7:

- Exercise 4 (hint: perhaps you should use direct proof in one direction, and contrapositive in the other),
- Exercise 12 (hint: just think of a number x that satisfies this property).

Additional Problem 1.

Let $x, y \in \mathbb{R}$. Prove that if x is rational and xy is irrational, then y is irrational.

Additional Problem 2.

(a) Let n be an arbitrary natural number. Use the pigeonhole principle to show that the number $n(n+1)(n+2)$ is divisible by 3.

Hint: Consider the remainders of the division of n, $(n+1)$, and $(n+2)$ by 3. What are the allowed values for the remainders from division by 3?

(b) Use the idea from part (a) to prove that the product of k consecutive natural numbers is divisible by k.

(c) Use (b) to explain why the product of k consecutive natural numbers is divisible by $k!$.

Hint: You have to explain why the product $n(n+1)(n+2)\cdots(n+k-1)$ of k consecutive natural numbers is divisible by 2, 3, ..., k, which would imply that it is divisible by $k!$.

(d) Use some elementary algebra and the result of part (a) to prove that $a^3 \equiv a \pmod 3$.

Additional Problem 3.

Suppose $a, b \in \mathbb{Z}$. Show that if $a^2 + b^2 = c^2$ for some integer $c \in \mathbb{Z}$, then a or b is even.

Hint: Let P, A, and B be respectively the statements $P = \{a^2 + b^2 = c^2\}$, $A = \{a \text{ is even}\}$, and $B = \{b \text{ is even}\}$. In this problem you have to show that $P \Rightarrow (A \vee B)$. Use contradiction, i.e., prove that if the negation of this statement is true, then you come to a contradiction. Use that

$$\sim\big[P \Rightarrow (A \vee B)\big] \;=\; \sim\big[(\sim P) \vee (A \vee B)\big] \;=\; \big[P \wedge \sim(A \vee B)\big] \;=\; \big[P \wedge (\sim A) \wedge (\sim B)\big] \,,$$

so that you have to assume that P, $\sim A$, and $\sim B$ are true, and come to a contradiction. The proof is very similar to the proof of Exercise 18 from Chapter 6 (one of the problems in this homework); see also the proof of Proposition at the top of p. 142 of the book.

Food for Thought Additional Problem (not to be turned in).

Read the solutions of:

- Hammack, Chapter 6: Exercises 3 and 5;
- Hammack, Chapter 7: Exercises 7, 9, 21, 25, and 27.