MATH 3413 Homework 8 Due Tue, Oct 24

Sec. 7.1: problem 40(b) (there is no need to write anything about part (a), but make sure that you understand it).

Sec. 7.2: problems 35, 37 (in these two problems use the extension of Theorem 1 on page 455 of the book).

Additional problem 1. Let $f(t)$ be a piecewise-continuous periodic function of period T.

(a) Show that

$$F(s) := \mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \ .$$

Hint: Apply directly the definition of Laplace transform:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt \ ,$$

then change the integration variable t to τ by the change $t = nT + \tau$, and use the fact that the periodicity of $f(t)$ implies that $f(nT+\tau) = f(\tau)$ for all τ.

(b) Use the formula derived in part (a) to find the Laplace transform of the function $f(t)$ you studied in problems 7.1/40 and 7.2/35.

Additional problem 2. Solve the following integrals:

$$\int_{-\infty}^{\infty} e^{-2t}\, \delta(t-3)\, dt \ , \quad \int_{-\infty}^{\infty} e^{-2t}\, \delta'(t-3)\, dt \ , \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-2t}\, \delta''(t-3)\, dt \ .$$

Additional problem 3.

(a) Find the transfer function $W(s)$ and the weight function $w(t)$ of the system described by the differential equation

$$x'' + 2x' + x = f(t) \ .$$

Assume that the initial conditions are $x(0) = 0$, $x'(0) = 0$.

(b) Use the convolution property to show that the solution of the initial value problem

$$x'' + 2x' + x = f(t) \ , \qquad x(0) = 0 \ , \quad x'(0) = 0 \tag{1}$$

can be written as

$$x(t) = \int_0^t \tau\, e^{-\tau} f(t-\tau)\, d\tau \ .$$

(c) Apply the formula for $x(t)$ obtained in part (b) to find the solution of the initial value problem (1) in the case $f(t) = e^{-t}$.

(d) The system described by the initial value problem (1) can be interpreted physically. Namely, $x(t)$ can be thought of as the position of a particle with mass $m = 1$ (the term x'') attached to a spring of spring constant $k = 1$ (the term x), in the presence of damping (the corresponding term is $2x'$ – it is important to notice that its coefficient is positive!) and an external driving force $f(t)$. In the case considered in part (c), the external force $f(t) = e^{-t}$ decreases with t, so one can expect that after long enough time the particle will slow down.

The initial coordinate of the particle is $x(0) = 0$. Find the maximum value of the coordinate $x(t)$ of the particle,

$$x_{\max} = \max_{t \geq 0} x(t) \ .$$

At which moment t^* does the particle have coordinate $x_{\max}$?

Additional problem 4.

(a) Use the formula for translation on the s-axis (page 458 of the book) to find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} \ .$$

(b) Using your result in (a) and the convolution property (pages 468–469), find

$$\mathcal{L}^{-1}\left\{\frac{1}{s} \cdot \frac{1}{(s+2)^2}\right\} \ .$$

(c) Apply the formula for translation on the t-axis (page 475) to obtain

$$\mathcal{L}^{-1}\left\{e^{-s} \frac{1}{(s+2)^2}\right\} \ .$$

(d) Use your results in parts (a)–(c) to show that the solution of the initial value problem

$$x'' + 4x' + 4x = 1 + \delta(t-1) \ , \qquad x(0) = 0 \ , \quad x'(0) = 0$$

is

$$x(t) = \frac{1}{4}\,[1 - e^{-2t} - 2t\,e^{-2t}] + (t-1)\,e^{-2(t-1)}\,u(t-1) \ .$$

The graph of $x(t)$ is shown in Figure 1. Note that the function $x(t)$ is continuous, but its slope has a jump at $t = 1$.

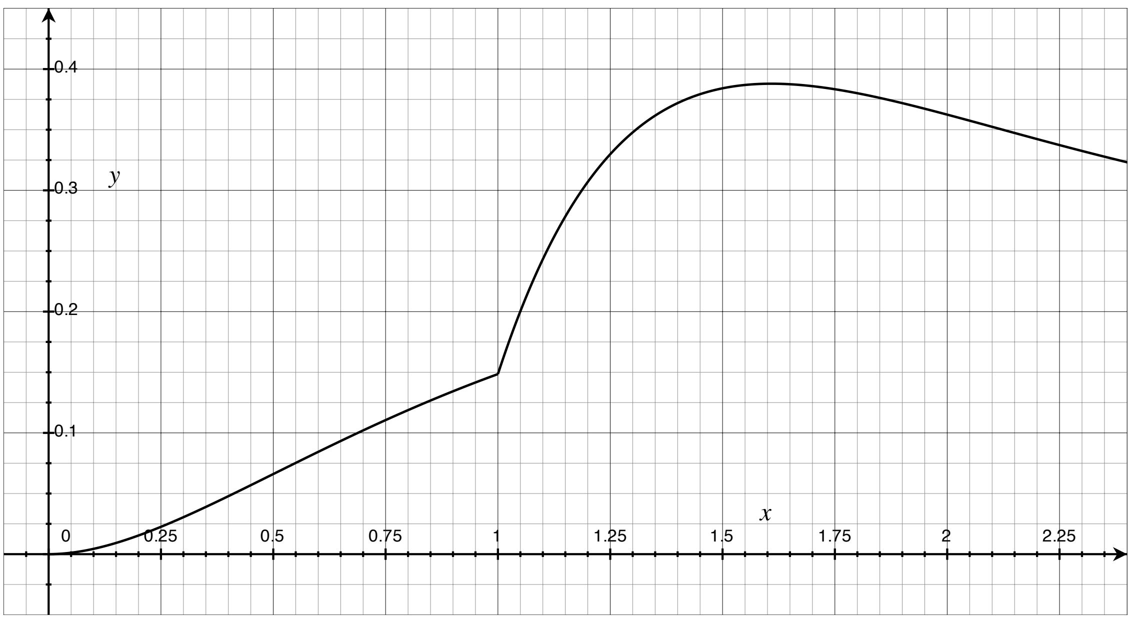

Figure 1: Graph of the function $x(t)$ from Additional problem 4.