MATH 3413 **Homework 8** **Due Thu, Apr 5, 2012**

Sec. 8.1: problem 4.

Sec. 9.1: problems 2, 9, 10, 11, 13, 27.

Additional problem 1. Let $\mathbb{R}^2$ stand for the vector space of all two-dimensional vectors, $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Let the inner product in $\mathbb{R}^2$ be given by

$$(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^{2} \sum_{j=1}^{2} u_i \, a_{ij} \, v_j \ ,$$

where $a_{11} = 2$, $a_{12} = a_{21} = 1$, $a_{22} = 4$.

Let $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ be a basis in $\mathbb{R}^2$, where $\mathbf{v}^{(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\mathbf{v}^{(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

(a) Check that $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ is an orthogonal basis, i.e., that the inner product of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ is zero: $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) = 0$.

(b) Find $(\mathbf{v}^{(1)}, \mathbf{v}^{(1)})$ and $(\mathbf{v}^{(2)}, \mathbf{v}^{(2)})$.

(c) If $\mathbf{u} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)}$, then find α_1 and α_2 by solving the system of linear equations for them coming from

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} + \alpha_2 \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} .$$

(d) Independently of part (c), if $\mathbf{u} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)}$, find α_1 and α_2 by using the formula

$$\alpha_j = \frac{\left(\mathbf{u}, \mathbf{v}^{(j)}\right)}{\left(\mathbf{v}^{(j)}, \mathbf{v}^{(j)}\right)} .$$

(e) The basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ is orthogonal, but not orthonormal. From the basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ construct an orthonormal basis $\{\widetilde{\mathbf{v}}^{(1)}, \widetilde{\mathbf{v}}^{(2)}\}$ of the form $\widetilde{\mathbf{v}}^{(i)} = \beta_i \, \mathbf{v}^{(i)}$, where β_i are appropriately chosen constants.

Additional problem 2. Using the properties of the Laplace transform, one can show that the solution of the initial value problem

$$\begin{aligned} x'(t) + 3x(t) &= f(t) \\ x(0) &= 0 \end{aligned} \tag{1}$$

can be written in the form

$$x(t) = \int_0^t f(\tau)\, \mathrm{e}^{3(\tau - t)}\, \mathrm{d}\tau \; . \tag{2}$$

In this problem you will check that the function $x(t)$ defined in (2) indeed satisfies the initial value problem (1). You will need to use the following formula:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\phi(t)}^{\psi(t)} F(\tau, t)\, \mathrm{d}\tau = F(\psi(t), t)\, \psi'(t) - F(\phi(t), t)\, \phi'(t) + \int_{\phi(t)}^{\psi(t)} \frac{\partial F(\tau, t)}{\partial t}\, \mathrm{d}\tau \; . \tag{3}$$

(a) Show that $x(t)$ given by (2) satisfies the initial condition in (1).

(b) If you represent $x(t)$ from (2) in the form $\displaystyle\int_{\phi(t)}^{\psi(t)} F(\tau, t)\, \mathrm{d}\tau$, then write explicitly the functions $F(\tau, t)$, $\phi(t)$, and $\psi(t)$.

(c) Find explicitly $F(\psi(t), t)\, \psi'(t)$ and $F(\phi(t), t)\, \phi'(t)$.

(d) Find explicitly $\displaystyle\int_{\phi(t)}^{\psi(t)} \frac{\partial F(\tau, t)}{\partial t}\, \mathrm{d}\tau$.

(e) Using your results from (b), (c), and (d), prove that $x(t)$ given by (2) satisfies the differential equation in (1).