

Sec. 8.1: problem 4.

Sec. 9.1: problems 2, 9, 10, 11, 13, 27.

Additional problem 1. Let \mathbb{R}^2 stand for the vector space of all two-dimensional vectors, $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$. Let the inner product in \mathbb{R}^2 be given by

$$(\mathbf{u}, \mathbf{v}) := \sum_{i=1}^2 \sum_{j=1}^2 u_i a_{ij} v_j ,$$

where $a_{11} = 2$, $a_{12} = a_{21} = 1$, $a_{22} = 4$.

Let $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ be a basis in \mathbb{R}^2 , where $\mathbf{v}^{(1)} = \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $\mathbf{v}^{(2)} = \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

- Check that $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ is an orthogonal basis, i.e., that the inner product of $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ is zero: $(\mathbf{v}^{(1)}, \mathbf{v}^{(2)}) = 0$.
- Find $(\mathbf{v}^{(1)}, \mathbf{v}^{(1)})$ and $(\mathbf{v}^{(2)}, \mathbf{v}^{(2)})$.
- If $\mathbf{u} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)}$, then find α_1 and α_2 by solving the system of linear equations for them coming from

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} + \alpha_2 \begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} .$$

- Independently of part (c), if $\mathbf{u} = \begin{pmatrix} 9 \\ -1 \end{pmatrix} = \alpha_1 \mathbf{v}^{(1)} + \alpha_2 \mathbf{v}^{(2)}$, find α_1 and α_2 by using the formula

$$\alpha_j = \frac{(\mathbf{u}, \mathbf{v}^{(j)})}{(\mathbf{v}^{(j)}, \mathbf{v}^{(j)})} .$$

- The basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ is orthogonal, but not orthonormal. From the basis $\{\mathbf{v}^{(1)}, \mathbf{v}^{(2)}\}$ construct an orthonormal basis $\{\tilde{\mathbf{v}}^{(1)}, \tilde{\mathbf{v}}^{(2)}\}$ of the form $\tilde{\mathbf{v}}^{(i)} = \beta_i \mathbf{v}^{(i)}$, where β_i are appropriately chosen constants.

Additional problem 2. Using the properties of the Laplace transform, one can show that the solution of the initial value problem

$$\begin{aligned} x'(t) + 3x(t) &= f(t) \\ x(0) &= 0 \end{aligned} \tag{1}$$

can be written in the form

$$x(t) = \int_0^t f(\tau) e^{3(\tau-t)} d\tau . \quad (2)$$

In this problem you will check that the function $x(t)$ defined in (2) indeed satisfies the initial value problem (1). You will need to use the following formula:

$$\frac{d}{dt} \int_{\phi(t)}^{\psi(t)} F(\tau, t) d\tau = F(\psi(t), t) \psi'(t) - F(\phi(t), t) \phi'(t) + \int_{\phi(t)}^{\psi(t)} \frac{\partial F(\tau, t)}{\partial t} d\tau . \quad (3)$$

- (a) Show that $x(t)$ given by (2) satisfies the initial condition in (1).
- (b) If you represent $x(t)$ from (2) in the form $\int_{\phi(t)}^{\psi(t)} F(\tau, t) d\tau$, then write explicitly the functions $F(\tau, t)$, $\phi(t)$, and $\psi(t)$.
- (c) Find explicitly $F(\psi(t), t) \psi'(t)$ and $F(\phi(t), t) \phi'(t)$.
- (d) Find explicitly $\int_{\phi(t)}^{\psi(t)} \frac{\partial F(\tau, t)}{\partial t} d\tau$.
- (e) Using your results from (b), (c), and (d), prove that $x(t)$ given by (2) satisfies the differential equation in (1).