MATH 3423 Homework 8 Due Mon, 11/7/16

Problem 1. Let $V_n(a, b; w(x))$ stand for the linear space of polynomials of degree no greater than n endowed with the inner product

$$\langle P, Q\rangle = \int_a^b P(x)\,Q(x)\,w(x)\,\mathrm{d}x\ .$$

We want to construct polynomials P_0, P_1, …, P_n satisfying the following conditions:

(i) the polynomial P_k is of degree k;

(ii) then coefficient of x^k in P_k is equal to 1 (such polynomials are called *monic*);

(iii) the polynomials P_0, P_1, P_2, …, P_n form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = \mathrm{e}^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^\infty x^k\,\mathrm{e}^{-x}\,\mathrm{d}x = k!$$

(where, by definition, $0! = 1$).

(a) Clearly, $P_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial P_1 of degree 1 that is orthogonal to P_0. Clearly, P_1 should have the form $P_1(x) = x+\alpha$, where α is a constant whose value you have to find. (The coefficient multiplying x is 1 because we want the polynomials P_k to be monic.)

(b) Find the only monic quadratic polynomial P_2 that is orthogonal to both P_0 and P_1. The polynomial P_2 should have the form $P_1(x) = x^2 + \beta x + \gamma$, where β and γ are constants whose values you have to find. (*Hint:* I obtained that $\gamma = 2$.)

(c) Show that the polynomial $Q(x) = x^2 + 3$ can be represented as a linear combination of the polynomials P_0, P_1 and P_2 as follows: $Q = P_2 + 4P_1 + 5P_0$.

(d) Show by direct integration that $\langle P_0, P_0\rangle = 1$, $\langle P_1, P_1\rangle = 1$, $\langle P_2, P_2\rangle = 4$.

(e) Find the orthogonal projection, $\mathrm{proj}_{P_0+2P_1}Q$, of the polynomial $Q(x) = x^2 + 3$ onto the " straight line"

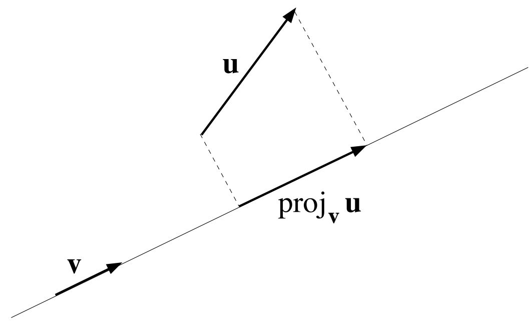
$$\ell := \{t(P_0 + 2P_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; \mathrm{e}^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If $\mathbf{u}$ and $\mathbf{v}$ are vectors in the inner product linear space V, then the orthogonal projection of the vector $\mathbf{u}$ onto the straight line in the direction of $\mathbf{v}$ is the vector

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\langle \mathbf{v}, \mathbf{v}\rangle}\,\mathbf{v}$$

– see the picture below.



(f) Finally, let $\widetilde{P}_k := \mu_k P_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\left\| \widetilde{P}_k \right\| := \sqrt{\langle \widetilde{P}_k, \widetilde{P}_k \rangle} \ ,$$

of the polynomial $\widetilde{P}_k$ is 1. Find the explicit expressions for $\widetilde{P}_0(x)$, $\widetilde{P}_1(x)$, and $\widetilde{P}_2(x)$.

Problem 2.

(a) Prove that $(\underline{\underline{A}}\,\underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$.

(b) Directly from the definition of orthogonality of matrices (for the case of Euclidean inner product), i.e., $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}}$, prove that the product of two orthogonal matrices is orthogonal.

Problem 3. Let the linear operator in the 2-dimensional vector space V with basis $\mathbf{f}_1$, $\mathbf{f}_2$, be defined by

$$\mathsf{A}\mathbf{f}_1 = -\mathbf{f}_1 + 4\mathbf{f}_2 \ ,$$

$$\mathsf{A}\mathbf{f}_2 = \mathbf{f}_1 + 2\mathbf{f}_2 \ .$$

(a) Write down the matrix $\underline{\underline{A}}$ of the linear operator A in the basis $\mathbf{f}_1$, $\mathbf{f}_2$.

(b) Compute the eigenvalues and the eigenvectors of this matrix.

Remark: In class we wrote $\underline{\underline{A}}$ and found that $\lambda_1 = -2$, $\lambda_2 = 3$, $\mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Here you only have to find an eigenvector $\mathbf{u}_2$. As you know, $\mathbf{u}_2$ is not uniquely defined; choose $\mathbf{u}_2$ it in such a way that its first component be equal to 1.

(c) Now you know that

$$\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{f}_2 \ ,$$

$$\mathbf{u}_2 = \mathbf{f}_1 + (?)\, \mathbf{f}_2 \ .$$

Express the original basis vectors $\mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in terms of the eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$. (Do not use any "canned" formulas, just do the obvious calculations.)

(d) Use the relations

$$\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{f}_2 \ ,$$
$$\mathbf{u}_2 = \mathbf{f}_1 + (?)\,\mathbf{f}_2$$

obtained in part (b), and the relations

$$\mathbf{f}_1 = \frac{4}{5}\,\mathbf{u}_1 + (?)\,\mathbf{u}_2 \ ,$$
$$\mathbf{f}_2 = (?)\,\mathbf{u}_1 + \frac{1}{5}\,\mathbf{u}_2$$

obtained in part (c), as well as the definition of the linear operator A in the statement of the problem (i.e., the action of A on the basis $\mathbf{f}_1$, $\mathbf{f}_2$), to express $\mathsf{A}\mathbf{u}_1$ and $\mathsf{A}\mathbf{u}_2$ in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$. At the end the result will be totally obvious, but I want to see your detailed calculations.

(e) Since the eigenvalues of the matrix $\underline{\underline{A}}$ are real and distinct, a theorem guarantees that the eigenvectors of the linear operator A form a basis of the linear space V. Let $\underline{\underline{\widetilde{A}}} = (\widetilde{a}_{ij})$ be the matrix of the linear operator A in the basis $\mathbf{u}_1$, $\mathbf{u}_2$, i.e., $\mathsf{A}\mathbf{u}_j = \sum_{i=1}^{2} \widetilde{a}_{ij}\,\mathbf{u}_i$. Find the entries $\widetilde{a}_{ij}$ of the matrix $\underline{\underline{\widetilde{A}}}$.

Remark: The result will be obvious, but I want to see all calculations that I am asking you to perform.

Problem 4. Determine the eigenvalues and eigenvectors of the matrix $\underline{\underline{A}} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$. How many linearly independent eigenvectors does it have?

Remark: This problem shows the trouble one may encounter in the case of repeated eigenvalues.

Problem 5. Express the coefficients of the characteristic polynomial, $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$, of the matrix $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of $\det A$ and $\operatorname{tr} A$.

Food for thought: The eigenvalues of an operator A should not depend on the choice of basis (because their definition did not require a choice of basis). On the other hand, the eigenvalues are roots of the characteristic equation $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$, which depends on the choice of basis in V (because in different bases the matrix $\underline{\underline{A}}$ of the linear operator A looks different). We know from the handout *Change of basis in a linear space* (linked at Lecture 26) that, if the change of basis is defined by the (invertible) matrix $\underline{\underline{A}}$, then the matrix $\underline{\underline{\widetilde{A}}}$ of the operator A in the new basis is related to the matrix $\underline{\underline{A}}$ of the operator in the old basis by $\underline{\underline{\widetilde{A}}} = \underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}$. This poses the question whether the characteristic polynomials $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}})$ and $\det(\underline{\underline{\widetilde{A}}} - \lambda \underline{\underline{I}})$ are the same (as functions of λ). Recalling the property $\det(\underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{A}})\det(\underline{\underline{B}})$ (which also implies that $\det(\underline{\underline{A}}^{-1}) = (\det\underline{\underline{A}})^{-1}$), we obtain

$$\det(\widetilde{\underline{\underline{A}}} - \lambda\underline{\underline{I}}) = \det(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1} - \lambda\underline{\underline{I}}) = \det(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1} - \lambda\underline{\underline{C}}\,\underline{\underline{I}}\,\underline{\underline{C}}^{-1})$$
$$= \det\left(\underline{\underline{C}}(\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{\underline{C}}^{-1}\right) = \det(\underline{\underline{C}})\det(\underline{\underline{A}} - \lambda\underline{\underline{I}})\det(\underline{\underline{C}}^{-1}) = \det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) \ ,$$

therefore the characteristic equations for the matrix of the operator A does not depend on the basis. One can also check that the determinant and the trace of the matrix of a linear operator A do not depend on the choice of basis. As you had to show in this problem, the characteristic equation has $\det\underline{\underline{A}}$ and $\mathrm{tr}\,\underline{\underline{A}}$ as coefficients. One can use the property of determinants to show that the determinant does not depend on the choice of basis:

$$\det(\widetilde{\underline{\underline{A}}}) = \det(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}) = \det\underline{\underline{C}}\ \det\underline{\underline{A}}\ \det(\underline{\underline{C}}^{-1}) = \det\underline{\underline{C}}\ \det\underline{\underline{A}}\,(\det\underline{\underline{C}})^{-1} = \det\underline{\underline{A}} \ .$$

As for the trace, one can easily prove that

$$\mathrm{tr}\left(\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\,\underline{\underline{D}}\right) = \mathrm{tr}\left(\underline{\underline{B}}\,\underline{\underline{C}}\,\underline{\underline{D}}\,\underline{\underline{A}}\right) = \mathrm{tr}\left(\underline{\underline{C}}\,\underline{\underline{D}}\,\underline{\underline{A}}\,\underline{\underline{B}}\right) = \mathrm{tr}\left(\underline{\underline{D}}\,\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\right)$$

(cyclic permutation of the product the matrices in the trace; analogous formula holds for the trace of the product of any number of matrices, not only four matrices as in this equality). Therefore

$$\mathrm{tr}\left(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}\right) = \mathrm{tr}\left(\underline{\underline{A}}\,\underline{\underline{C}}^{-1}\,\underline{\underline{C}}\right) = \mathrm{tr}\left(\underline{\underline{A}}\,\underline{\underline{I}}\right) = \mathrm{tr}\left(\underline{\underline{A}}\right) \ .$$

All this provides another proof that the eigenvalues of a 2×2 matrix does *not* depend on the choice of a basis.