

Problem 1. Let $V_n(a, b; w(x))$ stand for the linear space of polynomials of degree no greater than n endowed with the inner product

$$\langle P, Q \rangle = \int_a^b P(x) Q(x) w(x) dx .$$

We want to construct polynomials P_0, P_1, \dots, P_n satisfying the following conditions:

- (i) the polynomial P_k is of degree k ;
- (ii) then coefficient of x^k in P_k is equal to 1 (such polynomials are called *monic*);
- (iii) the polynomials $P_0, P_1, P_2, \dots, P_n$ form an orthogonal basis in the space of polynomials $V_n(0, \infty; w(x) = e^{-x})$.

In the solution of this problem the following identity will be handy:

$$\int_0^{\infty} x^k e^{-x} dx = k!$$

(where, by definition, $0! = 1$).

- (a) Clearly, $P_0(x) = 1$ for each $x \in [0, \infty)$. Find the only monic polynomial P_1 of degree 1 that is orthogonal to P_0 . Clearly, P_1 should have the form $P_1(x) = x + \alpha$, where α is a constant whose value you have to find. (The coefficient multiplying x is 1 because we want the polynomials P_k to be monic.)
- (b) Find the only monic quadratic polynomial P_2 that is orthogonal to both P_0 and P_1 . The polynomial P_2 should have the form $P_2(x) = x^2 + \beta x + \gamma$, where β and γ are constants whose values you have to find. (*Hint*: I obtained that $\gamma = 2$.)
- (c) Show that the polynomial $Q(x) = x^2 + 3$ can be represented as a linear combination of the polynomials P_0, P_1 and P_2 as follows: $Q = P_2 + 4P_1 + 5P_0$.
- (d) Show by direct integration that $\langle P_0, P_0 \rangle = 1$, $\langle P_1, P_1 \rangle = 1$, $\langle P_2, P_2 \rangle = 4$.
- (e) Find the orthogonal projection, $\text{proj}_{P_0+2P_1} Q$, of the polynomial $Q(x) = x^2 + 3$ onto the “straight line”

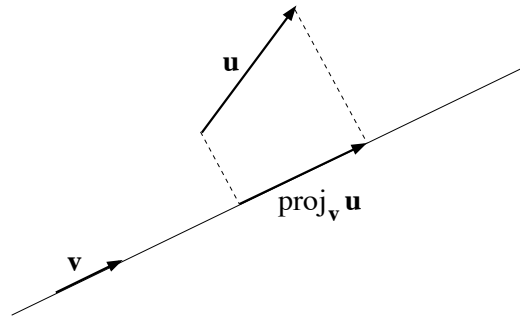
$$\ell := \{t(P_0 + 2P_1) \mid t \in \mathbb{R}\}$$

in the 3-dimensional inner product linear space $V_2(0, \infty; e^{-x})$. If you have solved part (c), then finding this orthogonal projection should be easy.

Hint: If \mathbf{u} and \mathbf{v} are vectors in the inner product linear space V , then the orthogonal projection of the vector \mathbf{u} onto the straight line in the direction of \mathbf{v} is the vector

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

– see the picture below.



(f) Finally, let $\tilde{P}_k := \mu_k P_k$, where $\mu_k > 0$ is a constant (depending on k) such that the norm,

$$\|\tilde{P}_k\| := \sqrt{\langle \tilde{P}_k, \tilde{P}_k \rangle},$$

of the polynomial \tilde{P}_k is 1. Find the explicit expressions for $\tilde{P}_0(x)$, $\tilde{P}_1(x)$, and $\tilde{P}_2(x)$.

Problem 2.

- (a) Prove that $(\underline{\underline{A}} \underline{\underline{B}})^T = \underline{\underline{B}}^T \underline{\underline{A}}^T$.
- (b) Directly from the definition of orthogonality of matrices (for the case of Euclidean inner product), i.e., $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{I}}$, prove that the product of two orthogonal matrices is orthogonal.

Problem 3. Let the linear operator in the 2-dimensional vector space V with basis $\mathbf{f}_1, \mathbf{f}_2$, be defined by

$$A\mathbf{f}_1 = -\mathbf{f}_1 + 4\mathbf{f}_2,$$

$$A\mathbf{f}_2 = \mathbf{f}_1 + 2\mathbf{f}_2.$$

- (a) Write down the matrix $\underline{\underline{A}}$ of the linear operator A in the basis $\mathbf{f}_1, \mathbf{f}_2$.
- (b) Compute the eigenvalues and the eigenvectors of this matrix.

Remark: In class we wrote $\underline{\underline{A}}$ and found that $\lambda_1 = -2, \lambda_2 = 3, \mathbf{u}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Here you only have to find an eigenvector \mathbf{u}_2 . As you know, \mathbf{u}_2 is not uniquely defined; choose \mathbf{u}_2 in such a way that its first component be equal to 1.

- (c) Now you know that

$$\mathbf{u}_1 = \mathbf{f}_1 - \mathbf{f}_2,$$

$$\mathbf{u}_2 = \mathbf{f}_1 + (?)\mathbf{f}_2.$$

Express the original basis vectors $\mathbf{f}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{f}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in terms of the eigenvectors \mathbf{u}_1 and \mathbf{u}_2 . (Do not use any “canned” formulas, just do the obvious calculations.)

(d) Use the relations

$$\begin{aligned}\mathbf{u}_1 &= \mathbf{f}_1 - \mathbf{f}_2, \\ \mathbf{u}_2 &= \mathbf{f}_1 + (?) \mathbf{f}_2\end{aligned}$$

obtained in part (b), and the relations

$$\begin{aligned}\mathbf{f}_1 &= \frac{4}{5} \mathbf{u}_1 + (?) \mathbf{u}_2, \\ \mathbf{f}_2 &= (?) \mathbf{u}_1 + \frac{1}{5} \mathbf{u}_2\end{aligned}$$

obtained in part (c), as well as the definition of the linear operator A in the statement of the problem (i.e., the action of A on the basis $\mathbf{f}_1, \mathbf{f}_2$), to express $A\mathbf{u}_1$ and $A\mathbf{u}_2$ in terms of \mathbf{u}_1 and \mathbf{u}_2 . At the end the result will be totally obvious, but I want to see your detailed calculations.

(e) Since the eigenvalues of the matrix $\underline{\underline{A}}$ are real and distinct, a theorem guarantees that the eigenvectors of the linear operator A form a basis of the linear space V . Let $\tilde{\underline{\underline{A}}} = (\tilde{a}_{ij})$ be the matrix of the linear operator A in the basis $\mathbf{u}_1, \mathbf{u}_2$, i.e., $A\mathbf{u}_j = \sum_{i=1}^2 \tilde{a}_{ij} \mathbf{u}_i$. Find the entries \tilde{a}_{ij} of the matrix $\tilde{\underline{\underline{A}}}$.

Remark: The result will be obvious, but I want to see all calculations that I am asking you to perform.

Problem 4. Determine the eigenvalues and eigenvectors of the matrix $\underline{\underline{A}} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$. How many linearly independent eigenvectors does it have?

Remark: This problem shows the trouble one may encounter in the case of repeated eigenvalues.

Problem 5. Express the coefficients of the characteristic polynomial, $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}})$, of the matrix $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in terms of $\det A$ and $\text{tr } A$.

Food for thought: The eigenvalues of an operator A should not depend on the choice of basis (because their definition did not require a choice of basis). On the other hand, the eigenvalues are roots of the characteristic equation $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0$, which depends on the choice of basis in V (because in different bases the matrix $\underline{\underline{A}}$ of the linear operator A looks different). We know from the handout *Change of basis in a linear space* (linked at Lecture 26) that, if the change of basis is defined by the (invertible) matrix $\underline{\underline{C}}$, then the matrix $\tilde{\underline{\underline{A}}}$ of the operator A in the new basis is related to the matrix $\underline{\underline{A}}$ of the operator in the old basis by $\tilde{\underline{\underline{A}}} = \underline{\underline{C}} \underline{\underline{A}} \underline{\underline{C}}^{-1}$. This poses the question whether the characteristic polynomials $\det(\underline{\underline{A}} - \lambda\underline{\underline{I}})$ and $\det(\tilde{\underline{\underline{A}}} - \lambda\underline{\underline{I}})$ are the same (as functions of λ). Recalling

the property $\det(\underline{A}\underline{B}) = \det(\underline{A})\det(\underline{B})$ (which also implies that $\det(\underline{A}^{-1}) = (\det \underline{A})^{-1}$), we obtain

$$\begin{aligned} \det(\tilde{\underline{A}} - \lambda\underline{I}) &= \det(\underline{C}\underline{A}\underline{C}^{-1} - \lambda\underline{I}) = \det(\underline{C}\underline{A}\underline{C}^{-1} - \lambda\underline{C}\underline{I}\underline{C}^{-1}) \\ &= \det(\underline{C}(\underline{A} - \lambda\underline{I})\underline{C}^{-1}) = \det(\underline{C})\det(\underline{A} - \lambda\underline{I})\det(\underline{C}^{-1}) = \det(\underline{A} - \lambda\underline{I}), \end{aligned}$$

therefore the characteristic equations for the matrix of the operator \mathbf{A} does not depend on the basis. One can also check that the determinant and the trace of the matrix of a linear operator \mathbf{A} do not depend on the choice of basis. As you had to show in this problem, the characteristic equation has $\det \underline{A}$ and $\text{tr} \underline{A}$ as coefficients. One can use the property of determinants to show that the determinant does not depend on the choice of basis:

$$\det(\tilde{\underline{A}}) = \det(\underline{C}\underline{A}\underline{C}^{-1}) = \det \underline{C} \det \underline{A} \det(\underline{C}^{-1}) = \det \underline{C} \det \underline{A} (\det \underline{C})^{-1} = \det \underline{A}.$$

As for the trace, one can easily prove that

$$\text{tr}(\underline{A}\underline{B}\underline{C}\underline{D}) = \text{tr}(\underline{B}\underline{C}\underline{D}\underline{A}) = \text{tr}(\underline{C}\underline{D}\underline{A}\underline{B}) = \text{tr}(\underline{D}\underline{A}\underline{B}\underline{C})$$

(cyclic permutation of the product the matrices in the trace; analogous formula holds for the trace of the product of any number of matrices, not only four matrices as in this equality). Therefore

$$\text{tr}(\underline{C}\underline{A}\underline{C}^{-1}) = \text{tr}(\underline{A}\underline{C}^{-1}\underline{C}) = \text{tr}(\underline{A}\underline{I}) = \text{tr}(\underline{A}).$$

All this provides another proof that the eigenvalues of a 2×2 matrix does *not* depend on the choice of a basis.