# MATH 3423 Homework 8 Due Thu, 11/7/19

**Problem 1.** Consider the following vectors in $\mathbb{R}^3$: $\mathbf{u} := \begin{pmatrix} 1 \\ 7 \\ -3 \end{pmatrix}$, $\mathbf{v} := \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$, and $\mathbf{w} := \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix}$. Directly from the definition, find out whether they are linearly dependent. Please write all your calculations in detail.

**Problem 2.** Let $V$ be a two-dimensional linear space, and $\mathsf{A} : V \to V$ be a linear operator acting on $V$. Let the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ form a basis in $V$, and the linear operator $\mathsf{A}$ acts on these two vectors as follows: $\mathsf{A}\mathbf{f}_1 = 2\mathbf{f}_1$, $\mathsf{A}\mathbf{f}_2 = 4\mathbf{f}_1 + \mathbf{f}_2$.

(a) Write down the matrix $\underline{\underline{A}} = (a_{ij})$ of the operator $\mathsf{A}$ in the basis $\mathbf{f}_1$, $\mathbf{f}_2$; recall that the entries of $\underline{\underline{A}}$ are defined by

$$\mathsf{A}\mathbf{f}_j =: \sum_{i=1}^{n} a_{ij}\,\mathbf{f}_i \ .$$

(b) If $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$, find $\mathsf{A}\mathbf{u}$ (expressed as a linear combination of the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$).

(c) Show that the vectors $\mathbf{u} = 3\mathbf{f}_1 - 5\mathbf{f}_2$ and $\mathbf{v} = 2\mathbf{f}_1 + \mathbf{f}_2$ are linearly independent, i.e., that if $\alpha\mathbf{u} + \beta\mathbf{v} = \mathbf{0}$ (where $\alpha, \beta \in \mathbb{R}$), then $\alpha = 0$ and $\beta = 0$.

**Problem 3.** Suppose that the linear operator $\mathsf{A} : \mathbb{R}^3 \to \mathbb{R}^3$ transforms $\mathbf{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ into $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ into $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, and $\mathbf{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ into $\mathbf{v}_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$. Find the matrix $\underline{\underline{A}}$ that corresponds to the operator $\mathsf{A}$.

*Hint:* You may use that, if $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is the standard basis in $\mathbb{R}^2$, then $\mathbf{u}_1 = \mathbf{e}_1 + \mathbf{e}_3$, $\mathsf{A}\mathbf{u}_1 = \mathbf{v}_1 = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$, which, by using the fact that $\mathsf{A}$ is a *linear* operator, is the same as $\mathsf{A}\mathbf{e}_1 + \mathsf{A}\mathbf{e}_2 = 2\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$; similarly, you will obtain two more equations: $\mathsf{A}\mathbf{e}_2 + \mathsf{A}\mathbf{e}_3 = \mathbf{e}_1 - \mathbf{e}_2$, $\mathsf{A}\mathbf{e}_1 + \mathsf{A}\mathbf{e}_2 =$ (something), from which you can express $\mathsf{A}\mathbf{e}_j$ ($j = 1, 2, 3$) in terms of the vectors $\mathbf{e}_i$ ($i = 1, 2, 3$).

**Problem 4.** Consider the linear space of polynomials of degree no greater than 3. Let us choose $\mathbf{f}_0 = 1$, $\mathbf{f}_1 = x$, $\mathbf{f}_2 = x^2$, $\mathbf{f}_3 = x^3$ to be a basis in this vector space, so that each polynomial $P(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$ can be written as a vector $\mathbf{p}$ in this space as

$$\mathbf{p} = \sum_{i=0}^{3} p_i\,\mathbf{f}_i \ .$$

(a) Let $\mathsf{D}$ be the differentiation operator, i.e., $\mathsf{D}\mathbf{p}$ is the polynomial $\frac{\mathrm{d}}{\mathrm{d}x}P(x)$. Find the matrix $\underline{\underline{D}}$ of $\mathsf{D}$ in the basis $\mathbf{f}_0$, $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$.

(b) Find the matrix of the operator $\mathsf{D}^2$ (where $\mathsf{D}^2 := \mathsf{D}\mathsf{D}$). You may use the fact that the matrix of a composition of the operators $\mathsf{A}$ and $\mathsf{B}$ is equal to the product of the matrices of $\mathsf{A}$ and $\mathsf{B}$.

(c) Find the matrices of $\mathsf{D}^k$ for $k = 3, 4, \ldots$. Do you get the zero operator $\mathsf{O}$ for some value of $k$? An operator $\mathsf{A}$ for which it happens that $\mathsf{A}^k = \mathsf{O}$ for some $k$ is called a *nilpotent* operator.

**Problem 5.** Consider the set of matrices

$$\mathcal{H} := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\} .$$

(a) Is the set $\mathcal{H}$ closed under matrix multiplication? (In other words, does the product of two matrices from $\mathcal{H}$ always belong to $\mathcal{H}$?) Justify your answer.

(b) Generally speaking, matrix multiplication is not commutative. Prove that if $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are matrices from $\mathcal{H}$, then $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{B}}\,\underline{\underline{A}}$.

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**Food for Thought Problem 1**[1] **[Change of basis in a linear space]**

Let $V$ be a linear space without any additional structure on it (i.e., there is no norm, inner product, or the concept of orthogonality). Consider the linear operator $\mathsf{A} : V \to V$ acting on $V$. Let $\mathbf{f}_1, \ldots, \mathbf{f}_n$ be a basis in $V$. One can define the matrix elements of $\mathsf{A}$ in this basis as follows:

$$\mathsf{A}\mathbf{f}_j =: \sum_{i=1}^{n} a_{ij}\,\mathbf{f}_i \ ,$$

so that the matrix of $\mathsf{A}$ in this basis is $\underline{\underline{A}} = (a_{ij})$ (see equation (7) on page 457).

(a) Let $\mathbf{u} = \sum_{j=1}^{n} u_j\,\mathbf{f}_j$. Find the components of $\mathsf{A}\mathbf{u}$ in the basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$; in other words, if $\mathbf{v} = \mathsf{A}\mathbf{u}$, find the numbers $v_i$ such that $\mathbf{v} = \sum_{i=1}^{n} v_i\,\mathbf{f}_i$.

Solution:

$$\mathbf{v} = \mathsf{A}\mathbf{u} = \mathsf{A}\sum_j u_j\,\mathbf{f}_j = \sum_j u_j\,\mathsf{A}\mathbf{f}_j$$

$$= \sum_j u_j \sum_i a_{ij}\,\mathbf{f}_i = \sum_i \left( \sum_j a_{ij}\,u_j \right) \mathbf{f}_i \ .$$

[1] You do not need to turn in the solution of this problem, but you are expected to understand it completely.

Comparing this with $\mathbf{v} = \sum_i v_i\, \mathbf{f}_i$, we see that

$$v_i = \sum_j a_{ij}\, u_j \ .$$

Recalling the definition of a multiplication of two matrices, we can interpret this equality as follows: if the vectors $\mathbf{u}$ and $\mathbf{v}$ are written as column vectors (i.e., as $n \times 1$ matrices, where $n = \dim V$), and $\underline{\underline{A}} = (a_{ij})$ is an $n \times n$ matrix, then the column vector $\mathbf{v} = \mathsf{A}\mathbf{u}$ is the product of the matrix $\underline{\underline{A}}$ and the column vector $\mathbf{u}$.

(b) Let $\mathsf{B}$ be another linear operator with matrix elements $(b_{ij})$ in the basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$. Find the matrix elements of the operator $\mathsf{AB}$ in the same basis. Here $\mathsf{AB}$ stands the composition of the operators in the sense that $(\mathsf{AB})\mathbf{u} := \mathsf{A}(\mathsf{B}\mathbf{u})$.

<u>Solution</u>:

$$(\mathsf{AB})\mathbf{u} = \mathsf{A}(\mathsf{B}\mathbf{u}) = \mathsf{A}\left(\mathsf{B}\sum_j u_j\, \mathbf{f}_j\right) = \mathsf{A}\left(\sum_j u_j\, \mathsf{B}\mathbf{f}_j\right) = \mathsf{A}\left(\sum_j u_j \sum_i b_{ij}\, \mathbf{f}_i\right)$$

$$= \mathsf{A}\left(\sum_{j,i} u_j\, b_{ij}\, \mathbf{f}_i\right) = \sum_{j,i} u_j\, b_{ij}\, \mathsf{A}\mathbf{f}_i = \sum_{j,i} u_j\, b_{ij} \sum_k a_{ki}\, \mathbf{f}_k = \sum_{j,i,k} u_j\, b_{ij}\, a_{ki}\, \mathbf{f}_k$$

$$= \sum_k \left[\sum_j \left(\sum_i a_{ki}\, b_{ij}\right) u_j\right] \mathbf{f}_k = \sum_k \left[\sum_j \left(\underline{\underline{A}}\,\underline{\underline{B}}\right)_{kj} u_j\right] \mathbf{f}_k \ ,$$

therefore the $k$th component of $(\mathsf{AB})\mathbf{u}$ is

$$[(\mathsf{AB})\mathbf{u}]_k = \sum_j \left(\underline{\underline{A}}\,\underline{\underline{B}}\right)_{kj} u_j \ .$$

This together with the result from part (a) allows us to conclude that the matrix of the linear operator $\mathsf{AB}$ is $\underline{\underline{A}}\,\underline{\underline{B}}$.

(c) Let $\mathsf{C} : V \to V$ be an *invertible* linear operator, i.e., a linear operator that has an inverse $\mathsf{C}^{-1}$, so that $\mathsf{C}(\mathsf{C}^{-1}) = \mathsf{C}^{-1}\mathsf{C} = \mathsf{I}$ (where $\mathsf{I}$ is the identity operator: $\mathsf{I}\mathbf{u} = \mathbf{u}$ for any $\mathbf{u} \in V$, the matrix of $\mathsf{I}$ in any basis is the unit matrix $\underline{\underline{I}} = (\delta_{ij})$). Let the operators $\mathsf{C}$ and $\mathsf{C}^{-1}$ have matrices $\underline{\underline{C}} = (c_{ij})$ and $\underline{\underline{D}} = (d_{ij})$, in the basis $\mathbf{f}_1, \ldots, \mathbf{f}_n$:

$$\mathsf{C}\mathbf{f}_j = \sum_{i=1}^{n} c_{ij}\, \mathbf{f}_i \ , \qquad \mathsf{C}^{-1}\mathbf{f}_j = \sum_{i=1}^{n} d_{ij}\, \mathbf{f}_i \ .$$

Show that the matrix $\underline{\underline{D}}$ corresponding to $\mathsf{C}^{-1}$ is equal to the inverse of the matrix $\underline{\underline{C}}$ corresponding to $\mathsf{C}$, i.e., $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$.

<u>Solution</u>: By the result of part (b), the operator $\mathsf{C}\,(\mathsf{C}^{-1})$ has matrix $\underline{\underline{C}}\,\underline{\underline{D}}$, so $\mathsf{C}\,(\mathsf{C}^{-1}) = \mathsf{I}$ implies that $\underline{\underline{C}}\,\underline{\underline{D}} = \underline{\underline{I}}$, i.e., that $\underline{\underline{D}} = \underline{\underline{C}}^{-1}$.

(d) Let us use the operators $\mathsf{C}$ and $\mathsf{C}^{-1}$ to define a new basis in $V$. Let the vectors of the new basis $\widetilde{\mathbf{f}}_1, \ldots, \widetilde{\mathbf{f}}_n$ be defined by

$$\mathbf{f}_j = \mathsf{C}\,\widetilde{\mathbf{f}}_j = \sum_{i=1}^{n} c_{ij}\,\widetilde{\mathbf{f}}_i \ ;$$

clearly, this implies that

$$\widetilde{\mathbf{f}}_j = \mathsf{C}^{-1}\,\mathbf{f}_j = \sum_{i=1}^{n} d_{ij}\,\mathbf{f}_i \left( = \sum_{i=1}^{n} (\underline{\underline{C}}^{-1})_{ij}\,\mathbf{f}_i \right) \ .$$

Prove that if $\widetilde{u}_j$ are the components of $\mathbf{u}$ in the new basis, i.e., $\mathbf{u} = \sum_{j=1}^{n} \widetilde{u}_j \widetilde{\mathbf{f}}_j$, then $\widetilde{u}_i = \sum_{j=1}^{n} c_{ij} u_j$.

Solution:

$$\mathbf{u} = \sum_j u_j\,\mathbf{f}_j = \sum_j u_j\,\mathsf{C}\,\widetilde{\mathbf{f}}_j = \sum_j u_j \sum_i c_{ij}\,\widetilde{\mathbf{f}}_i = \sum_i \left( \sum_j c_{ij} u_j \right) \widetilde{\mathbf{f}}_i$$

– compare with

$$\mathbf{u} = \sum_j \widetilde{u}_j\,\widetilde{\mathbf{f}}_j$$

to get

$$\widetilde{u}_i = \sum_j c_{ij}\,u_j \ .$$

(e) Now we want to find the matrix of the operator $\mathsf{A}$ in the new basis $\widetilde{\mathbf{f}}_1, \ldots, \widetilde{\mathbf{f}}_n$. Define the matrix elements $\widetilde{a}_{ij}$ of $\mathsf{A}$ in the new basis by

$$\mathsf{A}\widetilde{\mathbf{f}}_j =: \sum_{i=1}^{n} \widetilde{a}_{ij}\,\widetilde{\mathbf{f}}_i \ ,$$

and let $\underline{\underline{\widetilde{A}}} = (\widetilde{a}_{ij})$. Show that $\underline{\underline{\widetilde{A}}} = \underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}$.

Solution: Using the formulas from part (d) that connect the bases $\mathbf{f}_i$ and $\widetilde{\mathbf{f}}_i$, we obtain

$$\begin{aligned} \mathsf{A}\,\widetilde{\mathbf{f}}_j &= \mathsf{A} \sum_i d_{ij}\,\widetilde{\mathbf{f}}_i = \sum_i d_{ij}\,\mathsf{A}\,\widetilde{\mathbf{f}}_i \\ &= \sum_{i,k} d_{ij}\,a_{ki}\,\widetilde{\mathbf{f}}_k = \sum_{i,k,l} d_{ij}\,a_{ki}\,c_{lk}\,\widetilde{\mathbf{f}}_l = \sum_l \left( \sum_{i,k} c_{lk}\,a_{ki}\,d_{ij} \right) \widetilde{\mathbf{f}}_l \ . \end{aligned}$$

Comparing this with $\mathsf{A}\widetilde{\mathbf{f}}_j = \sum_l \widetilde{a}_{lj}\,\widetilde{\mathbf{f}}_l$, we see that

$$\widetilde{a}_{lj} = \sum_{i,k} c_{lk}\,a_{kl}\,d_{ij} \ ,$$

i.e., that $\underline{\underline{\widetilde{A}}} = \underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{C}}^{-1}$ (recall that $\underline{\underline{C}}^{-1} = (d_{ij})$).