# MATH 4073 Homework 8 Due Mon, 11/18/13

**Problem 1.** Recall that in the Taylor method of order 2 for solving the initial value problem $y'(t) = f(t, y(t))$, $y(a) = \alpha$, $t \in [a, b]$, the expressions are

$$w_{i+1} = w_i + h\,T^{(2)}(t_i, w_i)\ ,$$

where

$$T^{(2)}(t, w) = f(t, w) + \frac{h}{2}\,\frac{\mathrm{d}}{\mathrm{d}t} f(t, y(t))\bigg|_{y(t)=w} = f(t, w) + \frac{h}{2}\frac{\partial f}{\partial t}(t, w) + \frac{h}{2}\frac{\partial f}{\partial w}(t, w)\, f(t, w)\ .$$

The basic goal of the Runge-Kutta methods is to avoid computing derivatives of the function $f$. To achieve this goal, here you will develop a method that has the same local truncation error as the Taylor method of order 2.

(a) As a warm-up (not related to the rest of the problem), find the derivative $\frac{\mathrm{d}}{\mathrm{d}t} f\big(t, \mathrm{e}^{3t}\big)$. Please write the arguments of all functions! You can denote the partial derivatives of $f$ with respect to its first and second arguments as $\frac{\partial f}{\partial t}(t, w)$ and $\frac{\partial f}{\partial w}(t, w)$.

(b) We will replace $T^{(2)}(t, w)$ by an expression of the form

$$\frac{1}{2}\, f(t, w) + \frac{1}{2}\, f(t + \beta, w + \gamma)\ ,$$

where $\beta$ and $\gamma$ are some unknown constants. Expand the expression $f(t + \beta, w + \gamma)$ in a Taylor series around the point $(t, w)$, keeping only the terms linear in $\beta$ and $\gamma$ (i.e., neglecting all terms of the form $\beta^2$, $\beta\gamma$, $\gamma^2$ and all higher powers of $\beta$ and $\gamma$).

(c) Now equate the expression for $\frac{1}{2}\, f(t, w) + \frac{1}{2}\, f(t + \beta, w + \gamma)$ obtained in part (b) to $T^{(2)}(t, w)$ and compare the coefficients to find the constants $\beta$ and $\gamma$.

(d) Write the expression for $w_{i+1}$ that follows from your calculations in part (c).

**Problem 2.** In class we derived the 2-step Adams-Bashforth method, which is an explicit method in the sense that in the $i$th step we compute $w_{i+1}$ which is given as an explicit function of the values that we already know. The derivation started by integrating the ODE $y'(t) = f(t, y(t))$ from $t_i$ to $t_{i+1}$ to obtain

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t, y(t))\,\mathrm{d}t\ . \tag{1}$$

However, we cannot use this formula directly because the unknown function $y(t)$ is in the integrand $f(t, y(t))$ in the right-hand side of (1). The Adams-Bashforth approach is to replace $f(t, y(t))$ by a Lagrange interpolating polynomial that uses the values of $y(t)$ at the present time, $t_i$, and several previous times. In the 2-step Adams-Bashforth method we use the degree-1 Lagrange interpolating polynomial that uses the values $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$, where we have used the notations $y_{i-1} := y(t_{i-1})$, $y_i := y(t_i)$. Let this polynomial be $P_1(t)$ and the remainder term be $R_1(t)$, i.e.,

$$f(t, y(t)) = P_1(t) + R_1(t) \ .$$

The explicit expressions are

$$\begin{aligned} P_1(t) &= f(t_{i-1}, y_{i-1}) \frac{t - t_i}{t_{i-1} - t_i} + f(t_i, y_i) \frac{t - t_{i-1}}{t_i - t_{i-1}} \ , \\ R_1(t) &= \frac{f''(\xi(t), y(\xi(t)))}{2!} (t - t_i)(t - t_{i-1}) \ , \end{aligned}$$

where $\xi(t)$ is an unknown number in $[t_{i-1}, t_i]$, which depends on $t$. For the integral in the right-hand side of (1) we obtain

$$\begin{aligned} \int_{t_i}^{t_{i+1}} f(t, y(t)) \, \mathrm{d}t &\approx \int_{t_i}^{t_{i+1}} P_1(t) \, \mathrm{d}t \\ &= \int_{t_i}^{t_{i+1}} \left[ f(t_{i-1}, y_{i-1}) \frac{t - t_i}{t_{i-1} - t_i} + f(t_i, y_i) \frac{t - t_{i-1}}{t_i - t_{i-1}} \right] \mathrm{d}t \\ &= f(t_{i-1}, y_{i-1}) \int_{t_i}^{t_{i+1}} \frac{t - t_i}{t_{i-1} - t_i} \, \mathrm{d}t + f(t_i, y_i) \int_{t_i}^{t_{i+1}} \frac{t - t_{i-1}}{t_i - t_{i-1}} \, \mathrm{d}t \\ &= -\frac{h}{2} \, f(t_{i-1}, y_{i-1}) + \frac{3h}{2} \, f(t_i, y_i) \ . \end{aligned} \tag{2}$$

This together with (1) yields the 2-step Adams-Bashforth method (AB2)

$$w_{i+1} = w_i + h \left[ \frac{3}{2} \, f(t_i, w_i) - \frac{1}{2} f(t_{i-1}, w_{i-1}) \right] \ . \tag{3}$$

To obtain the error, we apply the so-called *Weighted Mean Value Theorem for Integrals*, which claims that, if $F(t)$ is a continuous function on $[a, b]$ and $G(t)$ does not change sign on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$\int_a^b F(t) \, G(t) \, \mathrm{d}t = F(c) \int_a^b G(t) \, \mathrm{d}t \ . \tag{4}$$

Using (4), we obtain, for some $c_i \in [t_{i-1}, t_i]$,

$$\begin{aligned} \int_{t_i}^{t_{i+1}} \left[ f(t, y(t)) - P_1(t) \right] \, \mathrm{d}t &= \int_{t_i}^{t_{i+1}} R_1(t) \, \mathrm{d}t \\ &= \int_{t_i}^{t_{i+1}} \frac{f(\xi(t), y(\xi(t)))}{2!} (t - t_i)(t - t_{i-1}) \, \mathrm{d}t \\ &= \frac{f(c_i, y(c_i))}{2} \int_{t_i}^{t_{i+1}} (t - t_i)(t - t_{i-1}) \, \mathrm{d}t \\ &= \frac{5}{12} \, f''(c_i, y(c_i)) \, h^3 = \frac{5}{12} \, y'''(c_i) \, h^3 \end{aligned} \tag{5}$$

(in the last step we used the differential equation to replace $f''(c_i, y(c_i))$ by $y'''(c_i)$). Using this, we can write the exact expression for $y_{i+1}$ as

$$y_{i+1} = y_i + h\left[\frac{3}{2}\, f(t_i, y_i) - \frac{1}{2} f(t_{i-1}, y_{i-1})\right] + \frac{5}{12}\, y'''(c_i, y(c_i))\, h^3 \ ,$$

therefore the local truncation error is

$$\tau_i(h) = \frac{y_{i+1} - y_i}{h} - \left[\frac{3}{2}\, f(t_i, y_i) - \frac{1}{2} f(t_{i-1}, y_{i-1})\right] = \frac{5}{12}\, y'''(c_i, y(c_i))\, h^2 \ ,$$

so that the local truncation error of the AB2 method is $O(h^2)$.

Below you have to follow a similar strategy to derive another method, but instead of taking the quadratic polynomial $P_2(t)$ which interpolates $f(t, y(t))$ (which is the integrand in the right-hand side of (1)) at the points $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$, you will use the polynomial $P_2(t)$ that interpolates $f(t, y(t))$ at the points $f(t_{i-1}, y_{i-1})$, $f(t_i, y_i)$, and $f(t_{i+1}, y_{i+1})$.

(a) Write down the quadratic Lagrange interpolating polynomial $P_2(t)$ that interpolates $f(t, y(t))$ at the points $f(t_{i-1}, y_{i-1})$, $f(t_i, y_i)$, and $f(t_{i+1}, y_{i+1})$. In other words, $P_2(t)$ is the (only) quadratic polynomial satisfying

$$P_2(t_{i-1}) = f(t_{i-1}, y_{i-1}) \ , \qquad P_2(t_i) = f(t_i, y_i) \ , \qquad P_2(t_{i+1}) = f(t_{i+1}, y_{i+1}) \ .$$

(b) Perform the calculations analogous to those in (2), but using the polynomial $P_2(t)$ as an approximate integrand in the right-hand side of (1). The following integrals will be useful:

$$\int_{t_i}^{t_{i+1}} (t - t_i)(t - t_{i+1})\, \mathrm{d}t = -\frac{h^3}{3} \ , \quad \int_{t_i}^{t_{i+1}} (t - t_{i-1})(t - t_{i+1})\, \mathrm{d}t = -\frac{2h^3}{3} \ , \quad \int_{t_i}^{t_{i+1}} (t - t_{i-1})(t - t_i)\, \mathrm{d}t = \frac{5h^3}{6} \ .$$

(c) Write the method that follows from your calculations in part (b) (analogously to (3)).

(d) Use the Weighted Mean Value Theorem for Integrals (4) and perform computations similar to (5) to derive an expression for the error, $\int_{t_i}^{t_{i+1}} [f(t, y(t)) - P_2(t)]\, \mathrm{d}t$. The following integral will be useful:

$$\int_{t_i}^{t_{i+1}} (t - t_{i-1})(t - t_i)(t - t_{i+1})\, \mathrm{d}t = -\frac{h^4}{4} \ .$$

(e) Find the local truncation error for the method derived in part (c).

(f) State clearly one advantage and one disadvantage of the method derived in part (c) in comparison with the AB2 method.