

Problem 1. Recall that in the Taylor method of order 2 for solving the initial value problem $y'(t) = f(t, y(t))$, $y(a) = \alpha$, $t \in [a, b]$, the expressions are

$$w_{i+1} = w_i + hT^{(2)}(t_i, w_i) ,$$

where

$$T^{(2)}(t, w) = f(t, w) + \frac{h}{2} \frac{d}{dt} f(t, y(t)) \Big|_{y(t)=w} = f(t, w) + \frac{h}{2} \frac{\partial f}{\partial t}(t, w) + \frac{h}{2} \frac{\partial f}{\partial w}(t, w) f(t, w) .$$

The basic goal of the Runge-Kutta methods is to avoid computing derivatives of the function f . To achieve this goal, here you will develop a method that has the same local truncation error as the Taylor method of order 2.

- (a) As a warm-up (not related to the rest of the problem), find the derivative $\frac{d}{dt} f(t, e^{3t})$. Please write the arguments of all functions! You can denote the partial derivatives of f with respect to its first and second arguments as $\frac{\partial f}{\partial t}(t, w)$ and $\frac{\partial f}{\partial w}(t, w)$.
- (b) We will replace $T^{(2)}(t, w)$ by an expression of the form

$$\frac{1}{2} f(t, w) + \frac{1}{2} f(t + \beta, w + \gamma) ,$$

where β and γ are some unknown constants. Expand the expression $f(t + \beta, w + \gamma)$ in a Taylor series around the point (t, w) , keeping only the terms linear in β and γ (i.e., neglecting all terms of the form β^2 , $\beta\gamma$, γ^2 and all higher powers of β and γ).

- (c) Now equate the expression for $\frac{1}{2} f(t, w) + \frac{1}{2} f(t + \beta, w + \gamma)$ obtained in part (b) to $T^{(2)}(t, w)$ and compare the coefficients to find the constants β and γ .
- (d) Write the expression for w_{i+1} that follows from your calculations in part (c).

Problem 2. In class we derived the 2-step Adams-Bashforth method, which is an explicit method in the sense that in the i th step we compute w_{i+1} which is given as an explicit function of the values that we already know. The derivation started by integrating the ODE $y'(t) = f(t, y(t))$ from t_i to t_{i+1} to obtain

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t, y(t)) dt . \tag{1}$$

However, we cannot use this formula directly because the unknown function $y(t)$ is in the integrand $f(t, y(t))$ in the right-hand side of (1). The Adams-Bashforth approach is to replace $f(t, y(t))$ by a Lagrange interpolating polynomial that uses the values of $y(t)$ at the present time, t_i , and several previous times. In the 2-step Adams-Bashforth method we use the degree-1 Lagrange interpolating polynomial that uses the values $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$, where we have used the

notations $y_{i-1} := y(t_{i-1})$, $y_i := y(t_i)$. Let this polynomial be $P_1(t)$ and the remainder term be $R_1(t)$, i.e.,

$$f(t, y(t)) = P_1(t) + R_1(t) .$$

The explicit expressions are

$$\begin{aligned} P_1(t) &= f(t_{i-1}, y_{i-1}) \frac{t - t_i}{t_{i-1} - t_i} + f(t_i, y_i) \frac{t - t_{i-1}}{t_i - t_{i-1}} , \\ R_1(t) &= \frac{f''(\xi(t), y(\xi(t)))}{2!} (t - t_i)(t - t_{i-1}) , \end{aligned}$$

where $\xi(t)$ is an unknown number in $[t_{i-1}, t_i]$, which depends on t . For the integral in the right-hand side of (1) we obtain

$$\begin{aligned} \int_{t_i}^{t_{i+1}} f(t, y(t)) dt &\approx \int_{t_i}^{t_{i+1}} P_1(t) dt \\ &= \int_{t_i}^{t_{i+1}} \left[f(t_{i-1}, y_{i-1}) \frac{t - t_i}{t_{i-1} - t_i} + f(t_i, y_i) \frac{t - t_{i-1}}{t_i - t_{i-1}} \right] dt \\ &= f(t_{i-1}, y_{i-1}) \int_{t_i}^{t_{i+1}} \frac{t - t_i}{t_{i-1} - t_i} dt + f(t_i, y_i) \int_{t_i}^{t_{i+1}} \frac{t - t_{i-1}}{t_i - t_{i-1}} dt \\ &= -\frac{h}{2} f(t_{i-1}, y_{i-1}) + \frac{3h}{2} f(t_i, y_i) . \end{aligned} \tag{2}$$

This together with (1) yields the 2-step Adams-Bashforth method (AB2)

$$w_{i+1} = w_i + h \left[\frac{3}{2} f(t_i, w_i) - \frac{1}{2} f(t_{i-1}, w_{i-1}) \right] . \tag{3}$$

To obtain the error, we apply the so-called *Weighted Mean Value Theorem for Integrals*, which claims that, if $F(t)$ is a continuous function on $[a, b]$ and $G(t)$ does not change sign on $[a, b]$, then there exists a number $c \in [a, b]$ such that

$$\int_a^b F(t) G(t) dt = F(c) \int_a^b G(t) dt . \tag{4}$$

Using (4), we obtain, for some $c_i \in [t_{i-1}, t_i]$,

$$\begin{aligned} \int_{t_i}^{t_{i+1}} [f(t, y(t)) - P_1(t)] dt &= \int_{t_i}^{t_{i+1}} R_1(t) dt \\ &= \int_{t_i}^{t_{i+1}} \frac{f(\xi(t), y(\xi(t)))}{2!} (t - t_i)(t - t_{i-1}) dt \\ &= \frac{f(c_i, y(c_i))}{2} \int_{t_i}^{t_{i+1}} (t - t_i)(t - t_{i-1}) dt \\ &= \frac{5}{12} f''(c_i, y(c_i)) h^3 = \frac{5}{12} y'''(c_i) h^3 \end{aligned} \tag{5}$$

(in the last step we used the differential equation to replace $f''(c_i, y(c_i))$ by $y'''(c_i)$). Using this, we can write the exact expression for y_{i+1} as

$$y_{i+1} = y_i + h \left[\frac{3}{2} f(t_i, y_i) - \frac{1}{2} f(t_{i-1}, y_{i-1}) \right] + \frac{5}{12} y'''(c_i, y(c_i)) h^3 ,$$

therefore the local truncation error is

$$\tau_i(h) = \frac{y_{i+1} - y_i}{h} - \left[\frac{3}{2} f(t_i, y_i) - \frac{1}{2} f(t_{i-1}, y_{i-1}) \right] = \frac{5}{12} y'''(c_i, y(c_i)) h^2 ,$$

so that the local truncation error of the AB2 method is $O(h^2)$.

Below you have to follow a similar strategy to derive another method, but instead of taking the quadratic polynomial $P_2(t)$ which interpolates $f(t, y(t))$ (which is the integrand in the right-hand side of (1)) at the points $f(t_{i-1}, y_{i-1})$ and $f(t_i, y_i)$, you will use the polynomial $P_2(t)$ that interpolates $f(t, y(t))$ at the points $f(t_{i-1}, y_{i-1})$, $f(t_i, y_i)$, and $f(t_{i+1}, y_{i+1})$.

- (a) Write down the quadratic Lagrange interpolating polynomial $P_2(t)$ that interpolates $f(t, y(t))$ at the points $f(t_{i-1}, y_{i-1})$, $f(t_i, y_i)$, and $f(t_{i+1}, y_{i+1})$. In other words, $P_2(t)$ is the (only) quadratic polynomial satisfying

$$P_2(t_{i-1}) = f(t_{i-1}, y_{i-1}) , \quad P_2(t_i) = f(t_i, y_i) , \quad P_2(t_{i+1}) = f(t_{i+1}, y_{i+1}) .$$

- (b) Perform the calculations analogous to those in (2), but using the polynomial $P_2(t)$ as an approximate integrand in the right-hand side of (1). The following integrals will be useful:

$$\int_{t_i}^{t_{i+1}} (t-t_i)(t-t_{i+1}) dt = -\frac{h^3}{3} , \quad \int_{t_i}^{t_{i+1}} (t-t_{i-1})(t-t_{i+1}) dt = -\frac{2h^3}{3} , \quad \int_{t_i}^{t_{i+1}} (t-t_{i-1})(t-t_i) dt = \frac{5h^3}{6} .$$

- (c) Write the method that follows from your calculations in part (b) (analogously to (3)).
- (d) Use the Weighted Mean Value Theorem for Integrals (4) and perform computations similar to (5) to derive an expression for the error, $\int_{t_i}^{t_{i+1}} [f(t, y(t)) - P_2(t)] dt$. The following integral will be useful:

$$\int_{t_i}^{t_{i+1}} (t-t_{i-1})(t-t_i)(t-t_{i+1}) dt = -\frac{h^4}{4} .$$

- (e) Find the local truncation error for the method derived in part (c).
- (f) State clearly one advantage and one disadvantage of the method derived in part (c) in comparison with the AB2 method.