

**Problem 1.** Consider the following BVP for the Laplace's equation:

$$\begin{aligned}\Delta u(x, y) &= 0, & x \in (0, 1), & \quad y \in (0, \infty), \\ u_y(x, 0) &= 0, \\ u_x(0, y) &= 0, \\ u(1, y) &= e^{-y}.\end{aligned}$$

Apply the Fourier cosine transform to show that

$$u(x, y) = \frac{2}{\pi} \int_0^\infty \frac{\cosh(\omega x) \cos(\omega y)}{(1 + \omega^2) \cosh \omega} d\omega.$$

Why do you have to use Fourier cosine transform (and not Fourier sine transform)?

*Hint:* This problem is somewhat related to the material in Section 10.6.2 of the book.

**Problem 2.** Consider a fourth-order linear differential operator

$$M := \frac{d^4}{dx^4}.$$

- (a) Let  $u$  and  $v$  be arbitrary smooth functions of one variable. Show that  $u Mv - v Mu$  is an exact differential, i.e., that it is equal to the derivative of some expression (involving the functions  $u$  and  $v$  and some of their derivatives).

*Hint:* Use integration by parts several times – for example,

$$u Mv = u v^{(4)} = \frac{d}{dx} [u v^{(3)}] - u' v^{(3)}.$$

- (b) Evaluate  $\int_0^1 [u Mv - v Mu] dx$  in terms of the boundary data for the functions  $u$  and  $v$  (i.e., in terms of  $u(0)$ ,  $v(0)$ ,  $u(1)$ ,  $v(1)$ ,  $u'(0)$ ,  $v'(0)$ ,  $u'(1)$ ,  $v'(1)$ , etc.).
- (c) Show that  $\int_0^1 [u Mv - v Mu] dx = 0$  if  $u$  and  $v$  are any two functions satisfying the boundary conditions

$$\begin{aligned}\phi(0) &= 0, & \phi(1) &= 0, \\ \frac{d\phi}{dx}(0) &= 0, & \frac{d^2\phi}{dx^2}(1) &= 0.\end{aligned}\tag{1}$$

This fact can be stated by saying that the differential operator  $M$  is self-adjoint in the space of functions satisfying the boundary conditions (1) (see the definition on p. 177).

(d) Give another example of boundary conditions such that

$$\int_0^1 [u Mv - v Mu] dx = 0 .$$

(e) For the eigenvalue problem

$$\frac{d^4\phi}{dx^4} + \lambda e^x \phi = 0$$

with the boundary conditions (1), show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weight function?

**Problem 3.** Consider the wave equation on the spatial interval  $(0, L)$  with a periodic driving force of frequency  $\omega$ :

$$\phi_{tt}(x, t) = c^2 \phi_{xx}(x, t) + g(x) e^{-i\omega t} , \quad x \in (0, L) , \quad t > 0 , \quad (2)$$

subjected to the homogeneous Dirichlet boundary conditions

$$\phi(0, t) = 0 , \quad \phi(L, t) = 0 . \quad (3)$$

(a) Show that a particular solution  $\phi(x, t) = u(x) e^{-i\omega t}$  is obtained if the function  $u$  satisfies a non-homogeneous *Helmholtz equation*,

$$\frac{d^2}{dx^2} u(x) + \beta^2 u(x) = f(x) , \quad (4)$$

for some positive constant,  $\beta > 0$ , and a function  $f$ . How are the constant  $\beta$  and the function  $f$  related to the constants  $c$  and  $\omega$  and the function  $g$  from (2)? What are the boundary conditions that  $u$  must satisfy?

(b) Suppose that you know the Green's function of the Helmholtz equation (4), which satisfies

$$\begin{aligned} \frac{d^2}{dx^2} G(x, \xi) + \beta^2 G(x, \xi) &= \delta(x - \xi) , \quad x \in (0, L) , \\ G(0, \xi) &= 0 , \quad G(L, \xi) = 0 . \end{aligned} \quad (5)$$

Express the solution of the boundary-value problem (2), (3) in terms of the Green's function  $G(x, \xi)$ .

(c) Write down the solution of the periodically driven wave equation (2) in terms of the functions found in part (b).

**Problem 4.** In this problem you will find the Green's function  $G(x, \xi)$  for the following boundary-value problem for the Helmholtz equation:

$$\begin{aligned} \frac{d^2}{dx^2}G(x, \xi) + \beta^2 G(x, \xi) &= \delta(x - \xi) , & x \in (0, L) , \\ G(0, \xi) &= 0 , & G(L, \xi) = 0 , \end{aligned} \tag{6}$$

by the method of variation of parameters (Section 9.3.2 in the book).

Here is a detailed plan. Consider the BVP

$$\begin{aligned} \frac{d^2 u}{dx^2}(x) + \beta^2 u(x) &= f(x) , & x \in (0, L) , \\ u(0) &= 0 , & u(L) = 0 . \end{aligned} \tag{7}$$

The differential operator in the left-hand side of the ODE in (7) is a particular case of the Sturm-Liouville operator,

$$\frac{d}{dx} \left( p(x) \frac{d}{dx} \right) + q(x) ,$$

for some particular choices of the functions  $p$  and  $q$  (which ones?). Look for a particular solution of the ODE  $u''(x) + \beta^2 u(x) = f(x)$  of the form

$$u(x) = v_1(x) u_1(x) + v_2(x) u_2(x) , \tag{8}$$

where  $u_1$  and  $u_2$  are two linearly independent solutions of  $u''(x) + \beta^2 u(x) = 0$ . You may take, for example,

$$u_1(x) := \sin \beta x , \quad u_2(x) := \cos \beta x$$

Impose on  $v_1'$  and  $v_2'$  the condition

$$u_1(x) v_1'(x) + u_2(x) v_2'(x) = 0 \tag{9}$$

and obtain that the ODE  $u''(x) + \beta^2 u(x) = f(x)$  is then satisfied if

$$p(x) u_1'(x) v_1'(x) + p(x) u_2'(x) v_2'(x) = f(x) . \tag{10}$$

Solve the system (9), (10) to find  $v_1'(x)$  and  $v_2'(x)$ , and then integrate to show that

$$v_1(x) = \frac{1}{\beta} \int_0^x \cos(\beta \xi) f(\xi) d\xi + C_1 , \quad v_2(x) = -\frac{1}{\beta} \int_0^x \sin(\beta \xi) f(\xi) d\xi + C_2 .$$

Plug these expressions in (8), and determine the constants from the fact that  $u$  must satisfy the boundary conditions in (7). I obtained the following expression:

$$\begin{aligned} u(x) &= \frac{1}{\beta} \left[ \cot(\beta L) \int_0^L \sin(\beta \xi) f(\xi) d\xi - \int_x^L \cos(\beta \xi) f(\xi) d\xi \right] \sin(\beta x) \\ &\quad - \frac{1}{\beta} \int_0^x \sin(\beta \xi) f(\xi) d\xi \cos(\beta x) . \end{aligned} \tag{11}$$

Finally, find an explicit expression for the function  $G(x, \xi)$  so that (11) can be written in the form  $u(x) = \int_0^L f(\xi) G(x, \xi) d\xi$ ; the explicit expression for  $G(x, \xi)$  will have different forms for  $x < \xi$  and for  $\xi < x$  (as in Equation 9.3.16 on page 388). You will have to split the integral  $\int_0^L$  as  $\int_0^x + \int_x^L$ , and use the trigonometric identity

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta .$$

**Problem 5.** Find the Green's function  $G(x, \xi)$  from Problem 4 by using the method of eigenfunction expansion (Section 9.3.3).

- (a) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} \frac{d^2 u}{dx^2}(x) + \beta^2 u(x) &= -\lambda u(x) , & x \in (0, L) , \\ u(0) = 0 , & \quad u(L) = 0 ; \end{aligned} \tag{12}$$

note that  $\sigma(x)$  has been set to be identically equal to 1. The ODE from (12) is nothing but the equation of the harmonic oscillator,

$$\frac{d^2 u}{dx^2}(x) + \left(\sqrt{\lambda + \beta^2}\right)^2 u(x) = 0 ,$$

– this fact can be used, together with the boundary conditions from (12) to obtain directly the eigenvalues  $\lambda_n$  and eigenfunctions  $\phi_n$  ( $n \in \mathbb{N}$ ).

- (b) Use  $\lambda_n$  and  $\phi_n$  to write  $G(x, \xi)$  explicitly.
- (c) Look at the concrete expressions for the eigenvalues  $\lambda_n$ . The constants in the problem,  $\beta$  and  $L$ , must satisfy some condition for the method of eigenfunction expansion to work. What is this condition? Discuss briefly.

**Problem 6.** Find the Green's function  $G(x, \xi)$  from Problem 4 directly, i.e., by using that  $\delta(x - \xi)$  is zero for  $x \neq \xi$ , and that  $G(x, \xi)$  must be continuous at  $x = \xi$  and its  $x$ -derivative must satisfy certain jump conditions there (see pages 395–396 of Section 9.3.4).

- (a) Let  $\xi$  be an arbitrary number in  $(0, L)$ . Write down the general solutions of the ODEs

$$\frac{d^2}{dx^2} G(x, \xi) + \beta^2 G(x, \xi) = 0 , \quad x \in (0, \xi) , \tag{13}$$

and

$$\frac{d^2}{dx^2} G(x, \xi) + \beta^2 G(x, \xi) = 0 , \quad x \in (\xi, L) . \tag{14}$$

The general solutions of (13) and (14) will have a total of four arbitrary constants.

(b) Impose the boundary conditions from (6) to show that

$$G(x, \xi) = \begin{cases} A \sin \beta x & \text{for } x < \xi, \\ B \sin \beta(L - x) & \text{for } \xi < x, \end{cases}$$

where  $A$  and  $B$  are some constants (still unknown).

- (c) Impose the continuity condition on  $G(x, \xi)$  at  $x = \xi$  to obtain a condition on the constants  $A$  and  $B$ .
- (d) Impose the jump condition on  $\frac{d}{dx}G(x, \xi)$  at  $x = \xi$  to obtain one more condition on the constants  $A$  and  $B$ .
- (e) Solve the conditions found in parts (c) and (d) to find the constants  $A$  and  $B$ , and write down  $G(x, \xi)$  explicitly.

*Hint:* You may need to use some trigonometric identity, i.e.,

$$\begin{aligned} \sin \beta \xi [\cot \beta \xi + \cot \beta(L - \xi)] &= \frac{\sin \beta(L - \xi) \cos \beta \xi + \cos \beta(L - \xi) \sin \beta \xi}{\sin \beta(L - \xi)} \\ &= \frac{\sin \beta L}{\sin \beta(L - \xi)}. \end{aligned}$$