MATH 4163 Homework 8-9 Due on Thu, Dec 6, in class

Problem 1. Consider the following BVP for the Laplace's equation:

$$\begin{aligned}
&\Delta u(x,y) = 0 \ , \qquad x \in (0,1) \ , \quad y \in (0,\infty) \ , \\
&u_y(x,0) = 0 \ , \\
&u_x(0,y) = 0 \ , \\
&u(1,y) = \mathrm{e}^{-y} \ .
\end{aligned}$$

Apply the Fourier cosine transform to show that

$$u(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\cosh(\omega x)\cos(\omega y)}{(1+\omega^2)\cosh\omega} \, \mathrm{d}\omega \ .$$

Why do you have to use Fourier cosine transform (and not Fourier sine transform)?

Hint: This problem is somewhat related to the material in Section 10.6.2 of the book.

Problem 2. Consider a fourth-order linear differential operator

$$M := \frac{\mathrm{d}^4}{\mathrm{d}x^4} \ .$$

(a) Let u and v be arbitrary smooth functions of one variable. Show that $u\,Mv - v\,Mu$ is an exact differential, i.e., that it is equal to the derivative of some expression (involving the functions u and v and some of their derivatives).

Hint: Use integration by parts several times – for example,

$$u\,Mv = u\,v^{(4)} = \frac{d}{\mathrm{d}x}\left[u\,v^{(3)}\right] - u'\,v^{(3)} \ .$$

(b) Evaluate $\displaystyle\int_0^1 [u\,Mv - v\,Mu] \, \mathrm{d}x$ in terms of the boundary data for the functions u and v (i.e., in terms of $u(0)$, $v(0)$, $u(1)$, $v(1)$, $u'(0)$, $v'(0)$, $u'(1)$, $v'(1)$, etc.).

(c) Show that $\displaystyle\int_0^1 [u\,Mv - v\,Mu] \, \mathrm{d}x = 0$ if u and v are any two functions satisfying the boundary conditions

$$\begin{aligned}
&\phi(0) = 0 \ , \qquad \phi(1) = 0 \ , \\
&\frac{\mathrm{d}\phi}{\mathrm{d}x}(0) = 0 \ , \qquad \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(1) = 0 \ .
\end{aligned} \tag{1}$$

This fact can be stated by saying that the differential operator M is self-adjoint in the space of functions satisfying the boundary conditions (1) (see the definition on p. 177).

(d) Give another example of boundary conditions such that

$$\int_0^1 [u\, Mv - v\, Mu]\, \mathrm{d}x = 0 \ .$$

(e) For the eigenvalue problem

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4} + \lambda \mathrm{e}^x \phi = 0$$

with the boundary conditions (1), show that the eigenfunctions corresponding to different eigenvalues are orthogonal. What is the weight function?

Problem 3. Consider the wave equation on the spatial interval $(0, L)$ with a periodic driving force of frequency ω:

$$\phi_{tt}(x,t) = c^2\, \phi_{xx}(xx) + g(x)\, \mathrm{e}^{-i\omega t} \ , \qquad x \in (0, L) \ , \quad t > 0 \ , \tag{2}$$

subjected to the homogeneous Dirichlet boundary conditions

$$\phi(0,t) = 0 \ , \qquad \phi(L,t) = 0 \ . \tag{3}$$

(a) Show that a particular solution $\phi(x,t) = u(x)\, \mathrm{e}^{-i\omega t}$ is obtained if the function u satisfies a non-homogeneous *Helmholtz equation*,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} u(x) + \beta^2 u(x) = f(x) \ , \tag{4}$$

for some positive constant, $\beta > 0$, and a function f. How are the constant β and the function f related to the constants c and ω and the function g from (2)? What are the boundary conditions that u must satisfy?

(b) Suppose that you know the Green's function of the Helmholtz equation (4), which satisfies

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}x^2} G(x,\xi) + \beta^2 G(x,\xi) = \delta(x-\xi) \ , \qquad x \in (0,L) \ , \\ &G(0,\xi) = 0 \ , \qquad G(L,\xi) = 0 \ . \end{aligned} \tag{5}$$

Express the solution of the boundary-value problem (2), (3) in terms of the Green's function $G(x,\xi)$.

(c) Write down the solution of the periodically driven wave equation (2) in terms of the functions found in part (b).

Problem 4. In this problem you will find the Green's function $G(x,\xi)$ for the following boundary-value problem for the Helmholtz equation:

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}x^2}G(x,\xi) + \beta^2 G(x,\xi) = \delta(x-\xi)\ , \qquad x \in (0,L)\ , \\ &G(0,\xi) = 0\ , \qquad G(L,\xi) = 0\ , \end{aligned} \tag{6}$$

by the method of variation of parameters (Section 9.3.2 in the book).

Here is a detailed plan. Consider the BVP

$$\begin{aligned} &\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x) + \beta^2 u(x) = f(x)\ , \qquad x \in (0,L)\ , \\ &u(0) = 0\ , \qquad u(L) = 0\ . \end{aligned} \tag{7}$$

The differential operator in the left-hand side of the ODE in (7) is a particular case of the Sturm-Liouville operator,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p(x)\frac{\mathrm{d}}{\mathrm{d}x}\right) + q(x)\ ,$$

for some particular choices of the functions p and q (which ones?). Look for a particular solution of the ODE $u''(x) + \beta^2 u(x) = f(x)$ of the form

$$u(x) = v_1(x)\, u_1(x) + v_2(x)\, u_2(x)\ , \tag{8}$$

where u_1 and u_2 are two linearly independent solutions of $u''(x) + \beta^2 u(x) = 0$. You may take, for example,

$$u_1(x) := \sin\beta x\ , \qquad u_2(x) := \cos\beta x$$

Impose on v_1' and v_2' the condition

$$u_1(x)\, v_1'(x) + u_2(x)\, v_2'(x) = 0 \tag{9}$$

and obtain that the ODE $u''(x) + \beta^2 u(x) = f(x)$ is then satisfied if

$$p(x)u_1'(x)\, v_1'(x) + p(x)u_2'(x)\, v_2'(x) = f(x)\ . \tag{10}$$

Solve the system (9), (10) to find $v_1'(x)$ and $v_2'(x)$, and then integrate to show that

$$v_1(x) = \frac{1}{\beta}\int_0^x \cos(\beta\xi)\, f(\xi)\,\mathrm{d}\xi + C_1\ , \qquad v_2(x) = -\frac{1}{\beta}\int_0^x \sin(\beta\xi)\, f(\xi)\,\mathrm{d}\xi + C_2\ .$$

Plug these expressions in (8), and determine the constants from the fact that u must satisfy the boundary conditions in (7). I obtained the following expression:

$$\begin{aligned} u(x) = {} & \frac{1}{\beta}\left[\cot(\beta L)\int_0^L \sin(\beta\xi)\, f(\xi)\,\mathrm{d}\xi - \int_x^L \cos(\beta\xi)\, f(\xi)\,\mathrm{d}\xi\right]\sin(\beta x) \\ & - \frac{1}{\beta}\int_0^x \sin(\beta\xi)\, f(\xi)\,\mathrm{d}\xi\ \cos(\beta x)\ . \end{aligned} \tag{11}$$

Finally, find an explicit expression for the function $G(x, \xi)$ so that (11) can be written in the form $u(x) = \displaystyle\int_0^L f(\xi)\, G(x, \xi)\, \mathrm{d}\xi$; the explicit expression for $G(x, \xi)$ will have different forms for $x < \xi$ and for $\xi < x$ (as in Equation 9.3.16 on page 388). You will have to split the integral $\int_0^L$ as $\int_0^x + \int_x^L$, and use the trigonometric identity

$$\sin(\alpha \pm \beta) = \sin\alpha\, \cos\beta \pm \cos\alpha\, \sin\beta\ .$$

Problem 5. Find the Green's function $G(x, \xi)$ from Problem 4 by using the method of eigenfunction expansion (Section 9.3.3).

(a) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$\begin{aligned} &\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x) + \beta^2 u(x) = -\lambda u(x)\ , \qquad x \in (0, L)\ , \\ &u(0) = 0\ , \qquad u(L) = 0\ ; \end{aligned} \tag{12}$$

note that $\sigma(x)$ has been set to be identically equal to 1. The ODE from (12) is nothing but the equation of the harmonic oscillator,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x) + \left(\sqrt{\lambda + \beta^2}\right)^2 u(x) = 0\ ,$$

– this fact can be used, together with the boundary conditions from (12) to obtain directly the eigenvalues λ_n and eigenfunctions ϕ_n ($n \in \mathbb{N}$).

(b) Use λ_n and ϕ_n to write $G(x, \xi)$ explicitly.

(c) Look at the concrete expressions for the eigenvalues λ_n. The constants in the problem, β and L, must satisfy some condition for the method of eigenfunction expansion to work. What is this condition? Discuss briefly.

Problem 6. Find the Green's function $G(x, \xi)$ from Problem 4 directly, i.e., by using that $\delta(x - \xi)$ is zero for $x \neq \xi$, and that $G(x, \xi)$ must be continuous at $x = \xi$ and its x-derivative must satisfy certain jump conditions there (see pages 395–396 of Section 9.3.4).

(a) Let ξ be an arbitrary number in $(0, L)$. Write down the general solutions of the ODEs

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} G(x, \xi) + \beta^2 G(x, \xi) = 0\ , \qquad x \in (0, \xi)\ , \tag{13}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} G(x, \xi) + \beta^2 G(x, \xi) = 0\ , \qquad x \in (\xi, L)\ . \tag{14}$$

The general solutions of (13) and (14) will have a total of four arbitrary constants.

(b) Impose the boundary conditions from (6) to show that

$$G(x,\xi) = \begin{cases} A\sin\beta x & \text{for } x < \xi \ , \\ B\sin\beta(L-x) & \text{for } \xi < x \ , \end{cases}$$

where A and B are some constants (still unknown).

(c) Impose the continuity condition on $G(x,\xi)$ at $x = \xi$ to obtain a condition on the constants A and B.

(d) Impose the jump condition on $\frac{\mathrm{d}}{\mathrm{d}x}G(x,\xi)$ at $x = \xi$ to obtain one more condition on the constants A and B.

(e) Solve the conditions found in parts (c) and (d) to find the constants A and B, and write down $G(x,\xi)$ explicitly.

Hint: You may need to use some trigonometric identity, i.e.,

$$\begin{aligned} \sin\beta\xi\ [\cot\beta\xi + \cot\beta(L-\xi)] &= \frac{\sin\beta(L-\xi)\ \cos\beta\xi + \cos\beta(L-\xi)\ \sin\beta\xi}{\sin\beta(L-\xi)} \\ &= \frac{\sin\beta L}{\sin\beta(L-\xi)} \ . \end{aligned}$$