MATH 4443 **Homework 8** **Due 3/29/18 (Thursday)**

Problem 1.

(a) Let $f : [-1, 1] \to \mathbb{R}$ be given by $f(x) = 2|x|$, and define $F(x) := \int_{-1}^{x} f$.

Find a piecewise algebraic formula for $F(x)$ for all $x \in [-1, 1]$. Looking at the formula, answer the following questions.

- Where is F continuous?
- Where is F differentiable?
- Where does $F'(x)$ equal $f(x)$?

(b) Repeat part (a) for the function $g : [-1, 1] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 1\ , & x \in [-1, 0), \\ 2\ , & x \in [0, 1]. \end{cases}$$

Problem 2.

(a) Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\int_{a}^{x} f = 0$ for all $x \in [a, b]$, then $f(x) = 0$ for all $x \in [a, b]$.

(b) Provide an example to show that the conclusion that f is identically zero on $[a, b]$ does not follow if f is not continuous.

Problem 3. Assume that $f_n \to f$ pointwise and $f_n' \to g$ uniformly on $[a, b]$. Assuming that each f_n' is continuous, we can apply the first part of the Fundamental Theorem of Calculus (i.e., Theorem 7.5.1(i)) to obtain

$$\int_{a}^{x} f_n' = f_n(x) - f_n(a) \qquad \text{for all } x \in [a, b]\ .$$

Show that $g(x) = f'(x)$.

Remark: This provides a simple proof of the Differentiable Limit Theorem (Theorem 6.3.1) under the additional assumption that the derivatives f_n' are continuous.

Problem 4. Let the function $L : (0, \infty)$ be defined by

$$L(x) = \int_{1}^{x} \frac{1}{t}\,\mathrm{d}t\ .$$

Pretend that you do not know anything about this function – the only thing that you are allowed to use in this problem is its definition.

(a) What is $L(1)$? Explain why L is differentiable and find $L'(x)$.

(b) Show that $L(xy) = L(x) + L(y)$.
Hint: Think of y as constant and differentiate $g(x) = L(xy)$.

(c) Show that $L\left(\frac{x}{y}\right) = L(x) - L(y)$.

(d) One can easily show that the range of L is the whole real line, and from the definition of L we see that L is strictly increasing, so that it is invertible. Let $E : \mathbb{R} \to (0, \infty)$ be the inverse function of L, i.e., $E \circ L : (0, \infty) \to (0, \infty)$ and $L \circ E : \mathbb{R} \to \mathbb{R}$ are the identities on $(0, \infty)$ and $\mathbb{R}$, respectively.

Use the properties of L to find $E(0)$ and to prove the identity $E(x + y) = E(x)\,E(y)$.

(e) Use the Inverse Function Theorem (Exercise 5.2.12) to find an expression for $E'(x)$.

(f) Let the sequence $(\gamma_n)_{n\in\mathbb{N}}$ be defined by

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n) \ .$$

Prove that the sequence (γ_n) converges. The constant

$$\gamma = \lim \gamma_n = 0.5772156649015328606065120900824024310421593359399 2\ldots$$

is called *Euler's constant* or *Euler-Mascheroni constant.*

(g) Show how consideration of the sequence $(\gamma_{2n} - \gamma_n)$ leads to the identity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \ .$$

Problem 5. Given a function $f : [a, b] \to \mathbb{R}$, define the *total variation* of f to be

$$V_a^b f = \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\} \ ,$$

where the supremum is taken over all partitions $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$.

(a) If f is continuously differentiable (i.e., f' exists and is a continuous function), use the Fundamental Theorem of Calculus to show that $V_a^b f \le \int_a^b |f'|$.

(b) Use the Mean Value Theorem to establish the reverse inequality and conclude that $V_a^b f = \int_a^b |f'|$.

Problem 6. In this problem you will prove the famous *Contraction Mapping Theorem* (often called *Banach Contraction Mapping Theorem*).

Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which there exists a constant c such that $0 < c < 1$, and

$$|f(x) - f(y)| \leq c\,|x - y| \ , \qquad \forall x, y \in \mathbb{R} \ .$$

This can also be stated as saying that f is Lipschitz with Lipschitz constant < 1. Geometrically speaking, this means that the distance between the images $f(x)$ and $f(y)$ is no greater than c times the distance between the original points x and y.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence $(y_n)_{n \in \mathbb{N}}$ iteratively by setting

$$y_{n+1} = f(y_n) \ .$$

Show that (y_n) is a Cauchy sequence. This allows you to conclude that (y_n) converges; let $y = \lim y_n$.

Hint: Show that $|y_{m+1} - y_{m+2}| \leq c^m |y_1 - y_2|$, then use the formula for geometric series to show that, for any $m < n$, $|y_m - y_n| \leq \dfrac{c^{m-1}}{1-c} |y_1 - y_2|$, and use this to prove that (y_n) is Cauchy.

(c) Prove that y (defined in part (b)) is a fixed point of the function f, i.e., that

$$f(y) = y \ .$$

(d) Prove that y (defined in part (b)) is the unique fixed point of the function f. This implies, in particular, that for any $x \in \mathbb{R}$, the sequence of iterates $(x, f(x), f(f(x)), \ldots)$ converges to y.

Food for Thought: Aksoy & Khamsi, Problem 7.19; Abbott, Exercises 7.5.2, 7.5.7.