MATH 5163 **Homework 8** **Due Wed, 4/9/14, 5 p.m.**

Problem 1. [The 1-dimensional wave equation]

In this problem you will rederive the expressions we obtained for the solution of the wave equation in one spatial dimension, using Fourier transform and Duhamel's principle. Please follow the steps below.

(a) Let $K(x,y,t)$ be the fundamental solution of the 1-dimensional wave equation, i.e.,

$$\begin{aligned} &K_{tt} - c^2 K_{xx} = 0 \ , \qquad x \in \mathbb{R} \ , \quad t > 0 \ , \\ &K(x,y,0) = 0 \ , \\ &K_t(x,y,0) = \delta(x-y) \ , \end{aligned}$$

where $y \in \mathbb{R}$ is a fixed value. Use Fourier transform in x,

$$\begin{aligned} \widehat{K}(\xi,y,t) &= \int_{\mathbb{R}} K(x,y,t)\, \mathrm{e}^{-\mathrm{i}\xi x}\, \mathrm{d}x \ , \\ K(x,y,t) &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{K}(\xi,y,t)\, \mathrm{e}^{\mathrm{i}\xi x}\, \mathrm{d}\xi \ , \end{aligned}$$

to write down and solve an initial-value problem for the Fourier transform $\widehat{K}(\xi,y,t)$ of $K(x,y,t)$.

(b) Perform inverse Fourier transform of $\widehat{K}(\xi,y,t)$ to derive an explicit expression for $K(x,y,t)$. You may find useful that

$$\left(\mathcal{F}\chi_{[-a,a]}\right)(\xi) = 2\,\frac{\sin a\xi}{\xi}$$

where $\chi_{[-a,a]}(x) = H(a-|x|)$ is the indicator function of the interval $[-a,a]$ for $a > 0$.

(c) Use the expression for $K(x,y,t)$ to write down the solution of the initial-value problem

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0 \ , \qquad x \in \mathbb{R} \ , \quad t > 0 \ , \\ &u(x,0) = 0 \ , \\ &u_t(x,0) = h(x) \ . \end{aligned}$$

Simplify the expression as much as possible (you should obtain a part of d'Alembert's formula).

(d) Use the expression for $K(x,y,t)$ to write down the solution of the initial-value problem

$$\begin{aligned} &v_{tt} - c^2 v_{xx} = 0 \ , \qquad x \in \mathbb{R} \ , \quad t > 0 \ , \\ &v(x,0) = g(x) \ , \\ &v_t(x,0) = 0 \ . \end{aligned}$$

In class we did this for the 3-dimensional wave equation, but the 1-dimensional case is completely analogous. Simplify the expression as much as possible (again, you should obtain a part of d'Alembert's formula).

(e) Use Duhamel's principle to write down the solution of the initial-value problem

$$\begin{aligned} & w_{tt} - c^2 w_{xx} = f(x,t) \ , \qquad x \in \mathbb{R} \ , \quad t > 0 \ , \\ & w(x,0) = 0 \ , \\ & w_t(x,0) = 0 \ . \end{aligned}$$

Again, simplify the expression as much as possible.

Problem 2. [Method of descent from the 2-D to the 1-D wave equation]
Use Hadamard's method of descent to derive the fundamental solution $K^{(1)}(x,t)$ of the wave equation in one spatial dimension from the fundamental solution of the 2-dimensional wave equation,

$$K^{(2)}(\vec{x},t) = \frac{1}{2\pi c} \frac{H(ct - |\vec{x}|)}{\sqrt{c^2t^2 - |\vec{x}|^2}} \ , \qquad \vec{x} = (x,y) \in \mathbb{R}^2 \ , \quad t > 0 \ ,$$

where H is the Heaviside function and $|\vec{x}| = \sqrt{x^2 + y^2}$.

Problem 3. [Uniqueness of the solution of the wave equation by energy method]
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^1 boundary $\partial\Omega$. Consider the initial boundary value problem

$$\begin{aligned} & u_{tt}(\mathbf{x},t) - c^2 \Delta u(\mathbf{x},t) = f(\mathbf{x},t) \ , \qquad (\mathbf{x},t) \in \Omega \times (0,\infty) \ , \\ & \text{BC} \quad \text{(Dirichlet, Neumann, or Robin)} \ , \\ & \text{IC:} \ \ u(\mathbf{x},0) = g(\mathbf{x}) \ , \ u_t(\mathbf{x},0) = h(\mathbf{x}) \ . \end{aligned} \tag{1}$$

The energy of the field $u(\mathbf{x},t)$ at time t is defined by

$$E(t) := \int_\Omega \left(\frac{1}{2} u_t^2 + \frac{c^2}{2} |\nabla u|^2 \right) \mathrm{d}\mathbf{x} \ .$$

(a) Use the PDE from (1) and the Green formula

$$\int_\Omega (\nabla\phi \cdot \nabla\psi + \phi\, \Delta\psi)\, \mathrm{d}\mathbf{x} = \oint_{\partial\Omega} \phi\, \nabla\psi \cdot \mathrm{d}\mathbf{S} \quad \left(= \oint_{\partial\Omega} \phi\, \frac{\partial\psi}{\partial\nu}\, \mathrm{d}S \right)$$

to find an expression for the rate of change $E'(t)$ of the energy of the physical system described by (1).

(b) Let $u(\mathbf{x},t)$ and $v(\mathbf{x},t)$ be solutions of the IBVP (1), and $w(\mathbf{x},t) := u(\mathbf{x},t) - v(\mathbf{x},t)$. Write down the IBVP satisfied by the function $w(\mathbf{x},t)$ for each of the following three boundary conditions:

$$\begin{aligned}
&u(\mathbf{x},t) = g(\mathbf{x},t) \ , \quad \mathbf{x} \in \partial\Omega \quad \text{(Dirichlet)} \ , \\
&\frac{\partial u}{\partial \nu}(\mathbf{x},t) = g(\mathbf{x},t) \ , \quad \mathbf{x} \in \partial\Omega \quad \text{(Neumann)} \ , \\
&\frac{\partial u}{\partial \nu}(\mathbf{x},t) + \alpha(\mathbf{x},t) u(\mathbf{x},t) = g(\mathbf{x},t) \ , \quad \mathbf{x} \in \partial\Omega \quad \text{(Robin)} \ ,
\end{aligned} \tag{2}$$

where α is a positive function and g is an arbitrary function (defined on $\partial\Omega \times (0,\infty)$).

(c) For each of the three types of boundary conditions (Dirichlet, Neumann, and Robin), show that the energy of the field w defined in part (b) is a non-increasing function of t. Use this to conclude that, if $w \in C^2(\Omega \times (0,\infty))$, then $w \equiv 0$. This shows the uniqueness of the solution of (1) in $C^2(\Omega \times (0,\infty))$.

Problem 4. [**"Sources of heat" coming from heat flux through the boundary**]

Imagine an infinite thin slab in $\mathbb{R}^3$ filling the space between the parallel planes $z = 0$ and $z = h$. Let the temperature in the slab be given by the function $u(x,y,z,t)$, where $(x,y) \in \mathbb{R}^2$, $z \in [0,h]$, $t > 0$. At the two boundaries (at $z = 0$ and at $z = h$) the slab exchanges heat with the surrounding atmosphere through convection. The temperature u_0 of the surrounding air is the same everywhere (on both sides of the slab). Assume that the initial temperature of the slab does not depend on z, but only on x and y. The temperature u in the slab is described by the initial-boundary value problem

$$\begin{aligned}
&c\rho\, u_t(\mathbf{x},t) = k\,\Delta u(\mathbf{x},t) \ , \qquad \mathbf{x} = (x,y,z) \in \mathbb{R}^2 \times [0,h] \ , \quad t > 0 \ , \\
&-\frac{\partial u}{\partial z}(x,y,0,t) + \alpha\left[u(x,y,0,t) - u_0\right] = 0 \ , \\
&\frac{\partial u}{\partial z}(x,y,h,t) + \alpha\left[u(x,y,h,t) - u_0\right] = 0 \ , \\
&u(\mathbf{x},0) = g(x,y) \ .
\end{aligned} \tag{3}$$

Here c, ρ, and k are positive constants characterizing the properties of the material of which the slab is made, and $\alpha > 0$ is a constant characterizing the rate of the heat exchange between the slab and the surrounding atmosphere.

If the slab is very thin, then one can ignore the variation of the temperature in z direction and consider it only as a function of x, y, and t; let $v(x,y,t)$ stand for the temperature in the slab in this approximation.

Although there are no sources of heat in the slab (and, correspondingly, no heat source term in the PDE in (3)), because of the heat flux through the planes at $z = 0$ and $z = h$, the slab will exchange heat with the surrounding atmosphere, so that the temperature $v(x,y,t)$ of the slab (in the approximation of very thin slab) will satisfy a heat equation with a term accounting for the sources of heat, i.e., an equation of the form

$$c\rho\, v_t(x,y,t) = k\,\Delta_\perp v(\mathbf{x},t) + \Psi(x,y,t)\ , \tag{4}$$

where $\Delta_\perp = \partial_{xx} + \partial_{yy}$ is the two-dimensional Laplacian. The density of the sources of heat – i.e., the function $\Psi(x,y,t)$ in the right-hand side of (4) – is an expression that involves the temperature $v(x,y,t)$. In this problem you will formulate an initial value problem for the temperature $v(x,y,t)$ in the thin slab. Please follow the steps below.

(a) Define the following differential operators:

$$\nabla_\perp := \mathbf{i}\,\partial_x + \mathbf{j}\,\partial_y\ , \qquad \Delta_\perp := \nabla_\perp\cdot\nabla_\perp = \partial_{xx} + \partial_{yy}\ ,$$

then clearly

$$\nabla = \nabla_\perp + \mathbf{k}\,\partial_z\ .$$

Show that

$$\Delta = \Delta_\perp + \partial_{zz}\ .$$

(b) Let $A \subset \mathbb{R}^2$ be an arbitrary open connected (i.e., consisting of "one piece") bounded domain in the (x,y)-plane, with boundary ∂A. Let $\boldsymbol{\nu}_A(x,y)$ be the outward unit normal vector to A at the point $(x,y) \in \partial A$. If $f : A \to \mathbb{R}$ is a scalar function, explain the origin of the identity

$$\iint_A \Delta f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \oint_{\partial A} \nabla_\perp f\cdot\boldsymbol{\nu}_A\,\mathrm{d}\ell = \oint_{\partial A} \frac{\partial f}{\partial \nu_A}(x,y)\,\mathrm{d}\ell\ ,$$

where $\mathrm{d}\ell$ is the line element of ∂A.

(c) Let $D := A \times [0,h] \subset \mathbb{R}^3$, where A is the two-dimensional domain from part (b). The boundary ∂D of D consists of three parts:

- top: $A \times \{h\} = \{(x,y,h) : (x,y) \in A\}$,
- bottom: $A \times \{0\} = \{(x,y,0) : (x,y) \in A\}$,
- side: $(\partial A) \times [0,h] = \{(x,y,z) : (x,y) \in \partial A, z \in [0,h]\}$.

The Divergence Theorem then implies

$$\begin{aligned}\iiint_D \Delta u(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \oint\!\!\!\oint_{\partial D} \nabla u(x,y,z,t)\cdot\mathrm{d}\mathbf{S}\\ &= \left(\iint_{\{\text{top}\}\cup\{\text{bottom}\}} + \iint_{\{\text{side}\}}\right)(\nabla_\perp + \mathbf{k}\,\partial_z)u(x,y,z,t)\cdot\mathrm{d}\mathbf{S}\ .\end{aligned}$$

Show in detail that

$$\iint_{\{\text{top}\}\cup\{\text{bottom}\}} (\nabla_\perp + \mathbf{k}\,\partial_z)u(x,y,z,t)\cdot\mathrm{d}\mathbf{S} = \iint_A \left[u_z(x,y,h,t) - u_z(x,y,0,t)\right]\mathrm{d}x\,\mathrm{d}y\ .$$

(d) Use the boundary conditions in (3) to rewrite the expression for the double integral $\iint_{\{\text{top}\}\cup\{\text{bottom}\}} (\nabla_\perp + \mathbf{k}\,\partial_z) u(x,y,z,t) \cdot \mathrm{d}\mathbf{S}$ obtained in part (c) in terms of the function u at the top and at the bottom of D. Then take the thickness h of the slab to zero and assume that the function u is independent of z, i.e., set $u(x,y,z,t) = v(x,y,t)$, to write the integral over $(\{\text{top}\} \cup \{\text{bottom}\})$ as a double integral over A of $[v(x,y,t) - u_0]$ (times some constants).

(e) Now rewrite $\iint_{\{\text{side}\}} (\nabla_\perp + \mathbf{k}\,\partial_z) u(x,y,z,t) \cdot \mathrm{d}\mathbf{S} = \iint_{(\partial A)\times[0,h]} (\nabla_\perp + \mathbf{k}\,\partial_z) u(x,y,z,t) \cdot \mathrm{d}\mathbf{S}$ under the assumption of very small thickness h and temperature u that is independent of z (i.e., $u(x,y,z,t) = v(x,y,t)$), as a double integral over A. The identity from part (b) will be useful.

(f) Integrate the PDE in (3) over the domain $D = A \times [0,h]$ and use your results from the previous parts of the problem to show that the PDE for the function $v(x,y,t)$ describing the temperature in a very thin slab is

$$c\rho\, v_t(x,y,t) = k\,\Delta_\perp v(\mathbf{x},t) - \frac{2\alpha k}{h}[v(x,y,t) - u_0]\ . \tag{5}$$

To check (just for yourself) that the units are OK, here are the units of the quantities involved (J = Joule, K = Kelvin):

$$[c] = \frac{\mathrm{J}}{\mathrm{kg\,K}}\ ,\quad [\rho] = \frac{\mathrm{kg}}{\mathrm{m}^3}\ ,\quad [k] = \frac{\mathrm{J}}{\mathrm{m\,s\,K}}\ ,\quad [v] = [u_0] = \mathrm{K}\ ,\quad [\alpha] = \frac{1}{\mathrm{m}}\ ,\quad [h] = \mathrm{m}$$

(g) To gain intuition, solve (5) in the extremely simplified situation when the temperature in the slab does not depend on x and y, but only on t; to this end, assume that the initial temperature in the slab is $v(x,y,0) = g_0 = \text{const}$. Write down an initial value problem for the ordinary differential equation that the function $w(t) := v(x,y,t)$ satisfies, and solve it. Does your solution look physically reasonable in the following aspects:

- the asymptotic behavior of the temperature in the slab $\lim_{t\to\infty} w(t)$;
- the dependence of $w(t)$ on the constant α proportional to the heat exchange between the slab and the surrounding air;
- the dependence of $w(t)$ on the thickness h of the slab?