MATH 5463 Homework 8 Due Thu, Apr 16

Problem 8.1/3 from Folland's book.

Hint: Use induction in k.

Additional problem 1. Is the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ uniformly continuous? Prove your claim.

Additional problem 2. Prove that, in general, translation is *not* continuous in the L^∞ norm. (Compare this result with Proposition 8.5.)

Hint: Here is an idea. Let $f \in L^\infty([-1,1], m)$ be defined as $f := \chi_{[-1,0)} - \chi_{[0,1]}$, and show that for $\epsilon = \frac{1}{2}$ there does not exist $g \in C_c([-1,1])$ such that $\|f - g\|_\infty < \frac{1}{2}$. To come to this conclusion, assume that such a function $g \in C_c([-1,1])$ exists, and show that this implies that $g(x) \in (-\infty, -\frac{1}{2}) \cup (\frac{1}{2}, \infty)$. Then use some of the basic theorems about continuous functions to show that the continuity of g implies the existence of an open interval in $[-1,1]$ on which $g(x) \in (-\frac{1}{2}, \frac{1}{2})$, and explain why this leads to contradiction.

Additional problem 3. Assume that for any $g \in L^1(\mathbb{R})$ and any $f \in C^1(\mathbb{R})$ the following holds:

$$(g * f')(x) = f(x+h) - f(x-h) ,$$

where h is a fixed real number. Show that $g = \chi_{[-h,h]}$.

Hint: Proposition 8.10 may serve as an inspiration.

Additional problem 4. Consider the space of real-valued functions on $(0,1)$ measurable with respect to m.

(a) Prove that if $\lim_{x\to\infty} |f(x)| = a > 0$, then $f \notin L^p((0,\infty), m)$ for any $p \in [1, \infty)$.

(b) Does the result of part (a) hold for $p = \infty$? Justify.

(c) Give an explicit example of a function $f \in L^p((0,\infty), m)$ (where p is any given value in $[1,\infty)$) such that $\lim_{x\to\infty} f(x) \neq 0$ even if we are allowed to set the values of f to be equal to zero on any Lebesgue-null set.

Additional problem 4. In this problem (taken from *Counterexamples in Analysis* by Gelbaum and Olmsted) you will show the existence of two metrics for the space $C([0,1])$ such that the complement of the unit ball in one of them is dense in the unit ball of the other.

Define the following two metrics on $C([0,1])$: $\rho_2(f,g) := \|f-g\|_2$, $\rho_\infty(f,g) := \|f-g\|_\infty$. Let $B_2 := \{f \in C([0,1]) : \rho_2(f,0) \leq 1\}$ and $B_\infty := \{f \in C([0,1]) : \rho_\infty(f,0) \leq 1\}$ be the unit balls (centered at the function identically equal to zero) in these two metrics.

(a) Show that $B_\infty \subset B_2$.

(b) Prove that the complement of B_∞ is dense in B_2. To this end, let $f \in B_2$ and $0 < \epsilon < 1$, and consider the following two cases separately:

- if $\|f\|_\infty > 1$, then ...;
- if $\|f\|_\infty \leq 1$, define the function $g \in C([0,1])$ by

$$g(x) := \begin{cases} 0 & \text{if } x \in [0, \frac{1}{2} - \frac{\epsilon^2}{9}] \cup [\frac{1}{2} + \frac{\epsilon^2}{9}, 1] \\ 3 & \text{if } x = \frac{1}{2} \\ \text{linear otherwise} & . \end{cases}$$

Then show that $f + g \notin B_\infty$ and $\rho_2(f, f+g) < \epsilon$.

Interpret your results to show that indeed you proved the desired result.