MATH 5763 Homework 8 Due Fri, 4/15/11

Problem 1. Let the function $F : \mathbb{R} \to [0,1]$ shown in Figure 1 be defined by $F(x) = 0$ for $x \le 0$, $F(x) = 1$ for $x \ge 1$, and

$$F(x) = \frac{1}{2^{j-1}} \qquad \text{for} \qquad \frac{1}{2^j} \le x < \frac{1}{2^{j-1}}\ , \quad j \in \{1, 2, 3, \dots\}$$

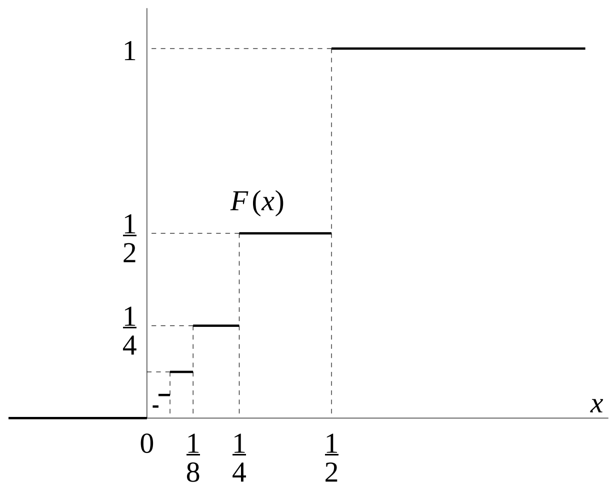

Figure 1: Graph of the function F.

Show that $\displaystyle\int_{\mathbb{R}} \mathrm{d}F(x) = 1\ ,\quad \int_{\mathbb{R}} x\,\mathrm{d}F(x) = \frac{1}{3}\ ,\quad \int_{\mathbb{R}} \ln x\,\mathrm{d}F(x) = -2\ln 2\ .$

Hint: You will need that, for $|q| < 1$, $\displaystyle\sum_{k=1}^{\infty} k\,q^{k-1} = \frac{\mathrm{d}}{\mathrm{d}q}\sum_{k=0}^{\infty} q^k = \frac{\mathrm{d}}{\mathrm{d}q}\frac{1}{1-q} = \frac{1}{(1-q)^2}.$

Problem 2. A system is made up of two components. We suppose that the lifetime (in years) of each component has an exponential distribution with parameter $\lambda = 2\,\mathrm{yr}^{-1}$, and that the components operate independently. When the system goes down, the two components are then immediately replaced by new ones. Consider the following three cases:

I. the two components are placed in series (so that both components must function for the system to work);

II. the two components are placed in parallel (so that a single operating component is sufficient for the system to function) and the two components operate at the same time);

III. the two components are placed in parallel, but only one component operates at a time, and the other component is in standby (i.e., ready to replace the first component when it fails).

Let $\{N_t : t \geq 0\}$ be the number of system failures in the interval $[0, t]$. Answer the following questions in each of the cases above.

(a) Is $\{N_t : t \geq 0\}$ a Poisson process? If it is, what is its rate? If it is not, justify, and determine the probability distribution of the inter-event times τ_j.

(b) What is the average time elapsed between two consecutive system failures? In two of the above three cases the answer is obvious (but I do want to see your calculations). Please discuss your results in these two cases.

Problem 3. In class we considered an M(λ)/G/1 queueing process $Q(t)$ where the arrival of customers is according to Poisson process of intensity λ (i.e., the inter-arrival times are i.i.d. random variables of type Exp(λ)), the service times are independent random variables with general (known) distribution, and one server. Let the service times S_n (where S_n is the time it takes to serve the nth customer) be independent random variables (which are also independent of the arrival times) with a common distribution that is known (i.e., you assume that you know the p.m.f. p_S of S_n if it is a discrete random variable, or the p.d.f. f_S of S_n if it is continuous).

Since such processes are difficult to deal with since they are not Markov, we defined a discrete stochastic process embedded in the queueing process $Q(t)$ that turns out to be Markov. Let D_n be the time of departure of the nth customer from the system, and $Y_n := Q(D^+)$ be the number of customers which the nth customer leaves behind after he/she leaves the queue. To show that $\{Y_n\}_{n=0}^{\infty}$ is a Markov process, let U_n be the number of customers arriving during the service time (of length S_{n+1}) of the $(n+1)$st customer. In class we showed that

$$Y_{n+1} = \begin{cases} U_n & \text{if } Y_n = 0 \ , \\ Y_n + U_n - 1 & \text{if } Y_n \geq 1 \ . \end{cases}$$

This can also be written as

$$Y_{n+1} = Y_n + U_n - 1 + h(Y_n) \ , \qquad \text{where } h(Y_n) := \begin{cases} 1 & \text{if } Y_n = 0 \ , \\ 0 & \text{if } Y_n \geq 1 \ . \end{cases}$$

Since the process of arrival is Poisson of intensity λ, the number of customers arriving during time interval of duration S_{n+1} is $U_n \sim \text{Poisson}(\lambda S_{n+1})$, from where it is clear that the discrete-time process $\{Y_n\}$ embedded in the continuous-time queueing process $\{Q(t)\}$ is a Markov process. We derived the transition probability matrix $\mathbf{P}$ of this discrete-time process:

$$\mathbf{P} = \begin{pmatrix} \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \delta_5 & \cdots \\ 0 & \delta_0 & \delta_1 & \delta_2 & \delta_3 & \delta_4 & \cdots \\ 0 & 0 & \delta_0 & \delta_1 & \delta_2 & \delta_3 & \cdots \\ 0 & 0 & 0 & \delta_0 & \delta_1 & \delta_2 & \cdots \end{pmatrix} , \qquad \text{where } \delta_j = \mathbb{E}\left[\frac{(\lambda S)^j}{j!}\, \mathrm{e}^{-\lambda S}\right] \ . \tag{1}$$

Define the *trafic intensity* (also called *traffic density*, or *offered load*) of the queue as

$$\rho := \frac{\text{expected service time}}{\text{expected interarrival time}} = \frac{\mathbb{E}[S]}{1/\lambda} = \lambda \mathbb{E}[S] \ .$$

One can show that for the embedded Markov chain $\{Y_n\}$ is positive recurrent if and only if $\rho < 1$; if $\rho = 1$, $\{Y_n\}$ is null-recurrent, and if $\rho > 1$, $\{Y_n\}$ is transient. If $\rho < 1$, $\{Y_n\}$ has a stationary distribution.

(a) Check that the matrix $\mathbf{P} = (p_{ij})_{i,j=0}^{\infty}$ defined by (1) is a stochastic matrix in general (i.e., for any distribution of the service time S).

Hint: This is very easy and requires almost no calculations, but you have to give me a *clear* argument, and to write explicitly which properties you use at each step.

(b) Compute δ_j in the case of $\mathrm{M}(\lambda)/\mathrm{M}(\mu)/1$ queue, i.e., when $S \sim \mathrm{Exp}(\mu)$ (in which case $\mathbb{E}[S] = \frac{1}{\mu}$).

(c) Compute δ_j if S is deterministic, taking value $\frac{1}{\mu}$ with probability 1. (We take S to be equal to $\frac{1}{\mu}$ so that $\mathbb{E}[S]$ be the same as in part (a).

(d) Compute δ_j if S is uniformly distributed on $[0, \frac{2}{\mu}]$ (so that, again, $\mathbb{E}[S] = \frac{1}{\mu}$).

Problem 4. Let $N = \{N_t : t \geq 0\}$ be a Poisson process with a constant rate λ. In the midterm exam you obtained many results about this process. You are allowed to use the information about Poisson processes given in the statement of the problem from the midterm, and can use in your answers to the questions below all results obtained in the midterm without deriving them again (but please write explicitly which results you use).

(a) Is N a wide-sense stationary process? Explain briefly.

(b) Is N a strong-sense stationary process? Why?

(c) Recall that the *probability mass function of order k* of a discrete state space stochastic process X was defined (on page 49 of the book) as

$$p(x_1, \ldots, x_k; t_1, \ldots, t_k) = \mathbb{P}(X_{t_1} = x_1, \ldots, X_{t_k} = x_k) \ .$$

Determine the probability mass functions of orders 1, 2, and 3, $p(i; r)$, $p(i, j; r, s)$, $p(i, j, k; r, s, t)$, of the Poisson process N assuming that $0 \leq r \leq s \leq t$, and that i, j, k are non-negative integers. Clearly, since the Poisson process is non-decreasing, $p(i, j; r, s)$ and $p(i, j, k; r, s, t)$ will be non-zero only if $i \leq j \leq k$.