Problem 1. Consider the linear constant coefficient system

$$x_1'(t) = x_1(t) + 2x_2(t)$$
$$x_2'(t) = 2x_1(t) + x_2(t) \ . \tag{1}$$

(a) Write the system (1) in the form $\mathbf{x}'(t) = \underline{\underline{A}}\mathbf{x}(t)$. Note that $\underline{\underline{A}}$ is a symmetric matrix.

(b) What does the general theory claim about the eigenvalues and eigenvectors of the matrix $\underline{\underline{A}}$?

(c) Find the eigenvectors and the normalized eigenvectors of the symmetric matrix $\underline{\underline{A}}$.

(d) Show that the eigenvectors and eigenvectors of $\underline{\underline{A}}$ found in (c) satisfy the properties that you predicted in (b).

(e) Write down the matrix $\underline{\underline{S}}$ whose columns are the normalized eigenvectors of $\underline{\underline{A}}$.

(f) Find $\underline{\underline{S}}^{-1}$. You can answer this question without doing any calculations, but please explain what properties you are using.

(g) Find $\underline{\underline{D}} = \underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}$ and compute $e^{t\underline{\underline{D}}}$.

(h) Use your results from parts (e)–(g) to compute $e^{t\underline{\underline{A}}}$.

(i) Use your result from part (h) to find the solution of the system (1) if $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

Problem 2. Solve the linear constant coefficient system (1) from the previous problem with initial condition $\mathbf{x}(0) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ by using that, if all eigenvalues λ_j of the matrix $\underline{\underline{A}}$ are distinct, then the general solution of the system $\mathbf{x}'(t) = \underline{\underline{A}}\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \sum_{j=1}^{n} C_j e^{\lambda_j t}\mathbf{u}_j \ ,$$

where $\mathbf{u}_j$ are the corresponding eigenvectors (not necessarily normalized).

Problem 3. As you know, one way to approximate a function f of one variable is to replace it by its tangent line at some point of interest, or by the "best fitting" parabola at this point (these approximations correspond to using the first- or second-order Taylor polynomial of the function f at this point). This type of approximation, however, works very well only near this point, and can be very inaccurate over an entire *interval*.

One way to approximate a function f (of one variable) on an entire interval is the following. Choose some class of functions $\mathcal{H}$, say all linear functions. Then look for a function h from this class $\mathcal{H}$ for which the "distance" between f and h is the smallest possible. The "distance" – which is

usually called "error" – can be defined in many different ways. If we want to approximate f by a function $h \in \mathcal{H}$ on the interval $[a, b]$, and we want $|f(x) - h(x)|$ to be small for all $x \in [a, b]$, then an appropriate definition for the "error" would be $E_\infty := \max\limits_{x \in [a,b]} |f(x) - h(x)|$. Another choice is to minimize $E_1 := \int_a^b |f(x) - h(x)| \, dx$, but the expressions for E_∞ and E_1 cause technical difficulties if one tries to use them in practice. The most convenient for numerical purposes expression for the error is

$$E_2 := \int_a^b [f(x) - h(x)]^2 \, dx \; ,$$

which we will use below. Incidentally, the cryptic notations E_∞, E_1, and E_2 are similar to the notations for the norms $\| \; \|_\infty$, $\| \; \|_1$, and $\| \; \|_2$.

In this problem you will find the best approximation of the function $f(x) = x^3$ by a linear function, $h_{\mu,\nu}(x) := \mu x + \nu$, over the interval $[0, 1]$ if the "error" is given by the integral

$$E_f(\mu, \nu) := \int_0^1 [f(x) - h_{\mu,\nu}(x)]^2 \, dx \; . \tag{2}$$

In other words, you have to choose the values of the constants μ and ν that minimize the error $E_f(\mu, \nu)$ given by (2).

Hint: Here is a useful fact: $\int_0^1 [x^3 - (\mu x + \nu)]^2 \, dx = \dfrac{1}{7} - \dfrac{2}{5}\mu + \dfrac{1}{3}\mu^2 - \dfrac{1}{2}\nu + \mu\nu + \nu^2 \; .$

Problem 4. In this problem you will solve in a geometric way the problem of finding the linear function that is "closest" to the function x^3 in the sense that it minimizes the "error" (2). Let $V_3(0, 1)$ stand for the linear space of polynomials on the interval $[0, 1]$ of degree no greater than 3, endowed with the inner product

$$\langle P, Q \rangle = \int_0^1 P(x) \, Q(x) \, dx \; , \qquad P \in V_3(0, 1) \; , \quad Q \in V_3(0, 1) \; .$$

Below we will use the "quantum mechanical notation"

$$\langle P | Q \rangle := \langle P, Q \rangle \; ,$$

where $\langle P |$ is the "bra-vector" corresponding to the "ket-vector" $|P\rangle$.

It is easy to show directly that the polynomials

$$N_0(x) = 1$$

$$N_1(x) = x - \frac{1}{2}$$

$$N_2(x) = x^2 - x + \frac{1}{6}$$

$$N_3(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}$$

form an orthogonal basis of the space $V_3(0, 1)$; the norms of these vectors are

$$\|N_0\| = \sqrt{\langle N_0, N_0 \rangle} = 1 \ , \qquad \|N_1\| = \frac{1}{\sqrt{12}} \ , \qquad \|N_2\| = \frac{1}{\sqrt{180}} \ , \qquad \|N_1\| = \frac{1}{\sqrt{2800}} \ ;$$

you do *not* need to do any of these calculations. This basis has the property that N_k is a polynomial of degree k.

Let $Q \in V_3(0, 1)$ be the polynomial

$$Q(x) = x^3 \ .$$

As in Problem 3, we want to find a linear function, i.e., a polynomial of degree no more than 1 that is "closest" to Q; such polynomials form a subspace of $V_3(0, 1)$ which we will denote by $V_1(0, 1)$:

$$V_1(0, 1) = \{ L : [0, 1] \to \mathbb{R} \mid L(x) = \mu x + \nu \, , \mu \in \mathbb{R}, \ \nu \in \mathbb{R} \} \ .$$

Since N_k is a polynomial of degree k, any polynomial of degree 1 – i.e., every $L \in V_1(0, 1)$ – is a linear combination of N_0 and N_1, so that we can write

$$V_1(0, 1) = \text{span} \{N_0, N_1\} = \{ L = \alpha N_0 + \beta N_1 \mid \alpha \in \mathbb{R}, \ \beta \in \mathbb{R} \} \ . \tag{3}$$

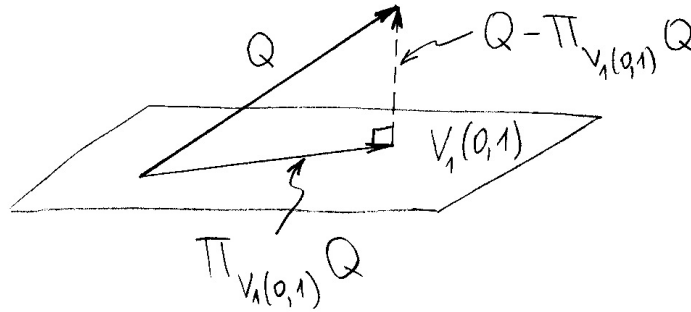Recall that, for any vector $P \in V_3(0, 1)$, the operator

$$\Pi_P := \frac{|P\rangle\langle P|}{\|P\|^2}$$

is the orthogonal projection of an arbitrary vector $Q \in V_3(0, 1)$ onto the direction of P:

$$\Pi_P|Q\rangle = \frac{|P\rangle\langle P|}{\|P\|^2} |Q\rangle = \frac{|P\rangle\langle P|Q\rangle}{\|P\|^2} \ .$$

Similarly, since the vectors N_0 and N_1 are orthogonal to one another, the orthogonal projection onto the plane $V_1(0, 1) = \text{span} \{N_0, N_1\}$ is given by the operator

$$\Pi_{V_1(0,1)} := \Pi_{N_0} + \Pi_{N_1} = \frac{|N_0\rangle\langle N_0|}{\|N_0\|^2} + \frac{|N_1\rangle\langle N_1|}{\|N_1\|^2} \ .$$

The projection of an arbitrary vector $|Q\rangle \in V_3(0, 1)$ onto the plane $V_1(0, 1)$ is, therefore, given by $\Pi_{V_1(0,1)}|Q\rangle \in V_1(0, 1)$. It can be shown that, among all vectors in $V_1(0, 1)$, the vector $\Pi_{V_1(0,1)}|Q\rangle$ is the one that is "closest" to $|Q\rangle \in V_3(0, 1)$ in the sense, that norm of the difference

$$|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle$$

is the smallest. Note that, if $|Q\rangle \in V_1(0, 1)$, then $|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle = 0$.

(a) Check that the vector $Q \in V_3(0,1)$ defined by $Q(x) = x^3$ can be written as a linear combination of the vectors from the orthogonal basis $\{N_0, N_1, N_2, N_3\}$ of $V_3(0,1)$ as

$$Q = \frac{1}{4}N_0 + \frac{9}{10}N_1 + \frac{3}{2}N_2 + N_3$$

(i.e., $x^3 = \frac{1}{4}N_0(x) + \frac{9}{10}N_1(x) + \frac{3}{2}N_2(x) + N_3(x)$).

(b) Show that the projection $\Pi_{V_1(0,1)}|Q\rangle$ is equal to $\frac{1}{4}N_0 + \frac{9}{10}N_1$.

Hint: This is very easy if you use the result of part (a) and the fact that the basis $\{N_0, N_1, N_2, N_3\}$ of $V_3(0,1)$ is orthogonal.

(c) Compare your answer to part (b) with the function $h_{\mu,\nu}$ that you found in Problem 3.

Problem 5. In this problem you will answer in a different way the same question as in Problems 3 and 4.

Looking at the figure in Problem 4, we see that, the shortest distance from "the end" of the vector Q to the plane $V_1(0,1)$ is accomplished if the difference $|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle$ is perpendicular to the plane $V_1(0,1)$. Since $\Pi_{V_1(0,1)}|Q\rangle$ belongs to $V_1(0,1)$ which was defined (recall (3)) as the span of the vectors N_0 and N_1, i.e., the set of all linear combinations of N_0 and N_1. Therefore, we have

$$\Pi_{V_1(0,1)}|Q\rangle = a|N_0\rangle + b|N_1\rangle .$$

Therefore, the vector

$$|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle = |Q\rangle - (a|N_0\rangle + b|N_1\rangle) = |Q\rangle - a|N_0\rangle - b|N_1\rangle \tag{4}$$

must be orthogonal to any vector from $V_1(0,1)$, which is equivalent to saying that it is orthogonal to each of the vectors $|N_0\rangle$ and $|N_1\rangle$ that "generate" the plane $V_1(0,1)$.

(a) Write down the conditions

$$\left(|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle\right) \perp N_0 ,$$

$$\left(|Q\rangle - \Pi_{V_1(0,1)}|Q\rangle\right) \perp N_1$$

for $Q(x) = x^3$ and N_0 and N_1 given in Problem 4, and derive a system of two equations for the constants a and b in (4). Your calculations will be greatly simplified if you use the representation of Q as a superposition of the vectors N_j that was derived in part (a) of Problem 4.

(b) Solve the system obtained in part (a). Compare your result for the vector $a|N_0\rangle + b|N_1\rangle \in V_1(0,1)$ from (4) with the vector $h_{\mu,\nu}$ obtained in Problem 3 and the vector $\Pi_{V_1(0,1)}|Q\rangle$ obtained in Problem 4.