MATH 4433 **Homework 9** **Due 4/18/17 (Tuesday)**

Abbott, Section 3.4:

Exercise 3.4.7 (page 106).

Abbott, Section 4.4:

Exercises 4.4.8, 4.4.11, 4.4.12 (pages 134, 135).

Remarks and hints:

- Exercise 4.4.11: It would be convenient to use Definition 3.2.1 and the characterization of continuity from Theorem 4.3.2(ii): $f : \mathbb{R} \to \mathbb{R}$ is continuous at a if and only if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $f(V_\delta(a)) \subseteq V_\varepsilon(f(a))$.

Abbott, Section 4.5:

Exercise 4.5.2 (pages 139, 140).

Additional Problem 1. Give an argument that shows that the image of a closed interval $[a, b]$ under a continuous function $f : [a, b] \to \mathbb{R}$ is a closed interval. Follow the following line of reasoning, filling out the missing pieces.

(a) The interval $[a, b]$ is closed and bounded, therefore it is .... According to Theorem ..., the image $f([a, b])$ of a ... set is ..., i.e., closed and ... (thanks to Theorem ...).

(b) Clearly, $f([a, b])$ is nonempty because ...; it is also bounded, therefore, by the Axiom of ..., it has an infimum m and ... M. According to the Extreme Value Theorem, ...

(c) According to the Intermediate Value Theorem, ..., therefore $f([a, b]) = [m, M]$.

Additional Problem 2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function s.t. $f(a) < 0 < f(b)$. Define the set $K \subseteq \mathbb{R}$ by

$$K := \{x \in [a, b] \,:\, f(x) \le 0\} \,.$$

Since K is nonempty ($a \in K$) and bounded above ($x \le b$ for any $x \in K$), it has a supremum $c = \sup K$. The fact that $f(b) > 0$ implies that $c < b$.

(a) Directly from the definition of continuity, prove that, if $f(z) > 0$ for some $z \in (a, b)$, then there exists $\delta > 0$ such that $f(V_\delta(z)) > 0$. (The same argument implies that if $f(z) < 0$, then $f(V_\delta(z)) < 0$ for some $\delta > 0$; the two arguments are so similar that you do not need to consider the $f(z) < 0$ case separately.)

(b) Use the fact proved in part (a) to show that $f(c) = 0$.

Additional Problem 3. Let the function $g : [a, b] \to \mathbb{R}$ be continuous, and let $g(x) \in [a, b]$ for every $x \in [a, b]$ (which can be written as $g([a, b]) \subseteq [a, b]$). Prove that g has at least one fixed point in $[a, b]$, i.e., one point $p \in [a, b]$ such that $g(p) = p$.

Pictorially, this has the following transparent geometric interpretation. Plot the graph of the function g, as well as the diagonal in the (x, y) plane, which is the graph of the function $d(x) = x$. Clearly, $d([a, b]) = [a, b]$. The point $(a, g(a))$ in the (x, y) plane must be on the segment of the straight line connecting the points (a, a) and (a, b) (why?). Similarly, the point $(b, g(b))$ in the (x, y) plane must be on the segment of the straight line connecting the points (b, a) and (b, b). It is intuitively clear that the graph of g – which connects the points $(a, g(a))$ and $(b, g(b))$ must cross the diagonal at least once. If this happens at the point $(p, g(p))$, p is a fixed point of g.