

Abbott, Section 3.4:

Exercise 3.4.7 (page 106).

Abbott, Section 4.4:

Exercises 4.4.8, 4.4.11, 4.4.12 (pages 134, 135).

Remarks and hints:

- Exercise 4.4.11: It would be convenient to use Definition 3.2.1 and the characterization of continuity from Theorem 4.3.2(ii): $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a if and only if $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $f(V_\delta(a)) \subseteq V_\varepsilon(f(a))$.

Abbott, Section 4.5:

Exercise 4.5.2 (pages 139, 140).

Additional Problem 1. Give an argument that shows that the image of a closed interval $[a, b]$ under a continuous function $f : [a, b] \rightarrow \mathbb{R}$ is a closed interval. Follow the following line of reasoning, filling out the missing pieces.

- The interval $[a, b]$ is closed and bounded, therefore it is According to Theorem ..., the image $f([a, b])$ of a ... set is ..., i.e., closed and ... (thanks to Theorem ...).
- Clearly, $f([a, b])$ is nonempty because ...; it is also bounded, therefore, by the Axiom of ..., it has an infimum m and ... M . According to the Extreme Value Theorem,
- According to the Intermediate Value Theorem, ..., therefore $f([a, b]) = [m, M]$.

Additional Problem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function s.t. $f(a) < 0 < f(b)$. Define the set $K \subseteq \mathbb{R}$ by

$$K := \{x \in [a, b] : f(x) \leq 0\} .$$

Since K is nonempty ($a \in K$) and bounded above ($x \leq b$ for any $x \in K$), it has a supremum $c = \sup K$. The fact that $f(b) > 0$ implies that $c < b$.

- Directly from the definition of continuity, prove that, if $f(z) > 0$ for some $z \in (a, b)$, then there exists $\delta > 0$ such that $f(V_\delta(z)) > 0$. (The same argument implies that if $f(z) < 0$, then $f(V_\delta(z)) < 0$ for some $\delta > 0$; the two arguments are so similar that you do not need to consider the $f(z) < 0$ case separately.)
- Use the fact proved in part (a) to show that $f(c) = 0$.

Additional Problem 3. Let the function $g : [a, b] \rightarrow \mathbb{R}$ be continuous, and let $g(x) \in [a, b]$ for every $x \in [a, b]$ (which can be written as $g([a, b]) \subseteq [a, b]$). Prove that g has at least one fixed point in $[a, b]$, i.e., one point $p \in [a, b]$ such that $g(p) = p$.

Pictorially, this has the following transparent geometric interpretation. Plot the graph of the function g , as well as the diagonal in the (x, y) plane, which is the graph of the function $d(x) = x$. Clearly, $d([a, b]) = [a, b]$. The point $(a, g(a))$ in the (x, y) plane must be on the segment of the straight line connecting the points (a, a) and (a, b) (why?). Similarly, the point $(b, g(b))$ in the (x, y) plane must be on the segment of the straight line connecting the points (b, a) and (b, b) . It is intuitively clear that the graph of g – which connects the points $(a, g(a))$ and $(b, g(b))$ must cross the diagonal at least once. If this happens at the point $(p, g(p))$, p is a fixed point of g .

