MATH 4443 **Homework 9** **Due 4/5/18 (Thursday)**

Problem 1. Demonstrate that $\rho(x,y) = |e^x - e^y|$ is a metric on $\mathbb{R}$.

Problem 2. Given a metric space (X,ρ), define a new metric on X by

$$\sigma(x,y) = \min\{\rho(x,y), 1\} \ .$$

(a) Show that σ is a metric on X.

Remark: Observe that X has a finite diameter in the σ metric.

(b) Show that $\lim_{n\to\infty} x_n = x$ in (X,ρ) if and only if $\lim_{n\to\infty} x_n = x$ in (X,σ).

(c) Show that the sequence (x_n) is Cauchy in (X,ρ) if and only if it is Cauchy in (X,σ). This means that (X,ρ) is complete if and only if (X,σ) is complete.

Problem 3. Two metrics ρ and σ on a set X are said to be *topologically equivalent* if for each $x \in X$ and each number $r > 0$, there is a number $s > 0$ (which in general depends on x and r) such that $B_s^\rho(x) \subset B_r^\sigma(x)$ and $B_s^\sigma(x) \subset B_r^\rho(x)$, where $B_r^\rho(x) := \{y \in X : \rho(x,y) < y\}$ is the open ball of radius r centered at x (and similarly for $B_s^\sigma(x)$, etc.).

(a) Recall that an open set A in a metric space (X,ρ) is defined as a set with the property that, if $x \in A$, then there exists a ball $B_r^\rho(x)$ that is entirely contained in A.

Prove that topologically equivalent metrics have the same open sets (which can be restated by saying that topologically equivalent metrics induce the same topology).

(b) Prove that topologically equivalent metrics have the same closed sets.

(c) Consider $\mathbb{R}$ with the two different metrics:

$$\rho(x,y) = |x-y| \ , \qquad \sigma(x,y) = |e^x - e^y| \ .$$

Prove that the metrics ρ and σ on X are topologically equivalent.

(d) The metric space (X,ρ) (defined in part (c)) is complete because it is a closed subset of the complete metric space $(\mathbb{R}, d)$ where $d(x,y) = |x-y|$ is the standard metric on $\mathbb{R}$. Consider the sequence $(x_n)_{n\in\mathbb{N}}$ given by $x_n = n$ in the metric space (X,σ) (defined in part (c)). Is (x_n) a Cauchy sequence in (X,σ)? Does it converge in (X,σ)?

(e) Discuss the meaning of your observation in part (d).

Problem 4. Two metrics ρ and σ on a set X are said to be *equivalent* (or *strongly equivalent*) if there exist constants $C_1 > 0$ and $C_2 > 0$ such that $C_1\rho(x,y) \le \sigma(x,y) \le C_2\rho(x,y)$ for all $x, y \in X$.

(a) Prove that equivalent metrics are topologically equivalent.

(b) Prove that equivalent metrics have the same Cauchy sequences.

(c) Give an example of topologically equivalent metrics that are not equivalent.

(d) [**Food for Thought only!**] Think about the meaning of the following statement:

> The continuity of a function $f : X \to Y$ (where (X, ρ) and (Y, τ) are metric spaces) is preserved if either ρ or τ is replaced by a topologically equivalent metric, but uniform continuity is preserved only if either ρ or τ is replaced by an equivalent metric.

Problem 5. Two norms, $\| \ \|$ and $\| \ \|'$, on the same vector space V are said to be *equivalent* if there exist positive constants C_1 and C_2 such that $C_1\|\mathbf{u}\| \le \|\mathbf{u}\|' \le C_2\|\mathbf{u}\|$ for any $\mathbf{u} \in V$. Consider the vector space $\mathbb{R}^n$ with the following norms defined on it:

$$\|\mathbf{u}\|_1 := \sum_{j=1}^{n} |u_j| \ , \qquad \|\mathbf{u}\|_2 := \left(\sum_{j=1}^{n} |u_j|^2 \right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty := \max_{1 \le j \le n} |u_j| \ .$$

(a) Prove that the norms $\| \ \|_1$ and $\| \ \|_\infty$ on $\mathbb{R}^n$ are equivalent.

(b) Directly from the definition of equivalence of norms, prove that if the norms $\| \ \|$ and $\| \ \|'$ are equivalent and the norms $\| \ \|'$ and $\| \ \|''$ are equivalent, then the norms $\| \ \|$ and $\| \ \|''$ are equivalent.

Problem 6. Many theorems that hold in finite-dimensional spaces are not true in infinite-dimensional spaces. One can think of the real infinite-dimensional space $\mathbb{R}^\infty$ as the space of infinite sequences: $\mathbf{u} = (u_1, u_2, u_3, \ldots)$, where u_j are real numbers ($j \in \mathbb{N} := \{1, 2, 3, \ldots\}$). In this space we can define the norms $\| \ \|_1$, $\| \ \|_2$, and $\| \ \|_\infty$ as usual:

$$\|\mathbf{u}\|_1 := \sum_{j \in \mathbb{N}} |u_j| \ , \qquad \|\mathbf{u}\|_2 := \left(\sum_{j \in \mathbb{N}} |u_j|^2 \right)^{1/2} \ , \qquad \|\mathbf{u}\|_\infty := \sup_{j \in \mathbb{N}} |u_j| \ .$$

The notations ℓ^1, ℓ^2, and ℓ^∞ are sometimes used for the spaces of infinite sequences whose $\|\mathbf{u}\|_1$, $\|\mathbf{u}\|_2$, or $\|\mathbf{u}\|_\infty$, are finite:

$$\begin{aligned} \ell^1 &:= \{\mathbf{u} \in \mathbb{R}^\infty \ : \ \|\mathbf{u}\|_1 < \infty\} \ , \\ \ell^2 &:= \{\mathbf{u} \in \mathbb{R}^\infty \ : \ \|\mathbf{u}\|_2 < \infty\} \ , \\ \ell^\infty &:= \{\mathbf{u} \in \mathbb{R}^\infty \ : \ \|\mathbf{u}\|_\infty < \infty\} \ . \end{aligned}$$

One can show that $\ell^1 \subseteq \ell^2 \subseteq \ell^\infty$ (you do *not* need to do this here). In this problem you will give examples showing that these inclusions are strict, i.e., that there exist vectors that are in ℓ^2 but not in ℓ^1, and there exist vectors that are in ℓ^∞ but not in ℓ^2.

(a) Give an explicit example of a sequence $\mathbf{v}$ such that $\|\mathbf{v}\|_\infty < \infty$, but $\|\mathbf{v}\|_2$ is infinite.

(b) Give an explicit example of a sequence $\mathbf{w}$ such that $\|\mathbf{w}\|_2 < \infty$, but $\|\mathbf{w}\|_1$ is infinite.
Hint: Think how you can use the following facts:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6} \,, \qquad \sum_{j=1}^{\infty} \frac{1}{j} = \infty \,.$$

Problem 7. In this problem you will prove the famous *Contraction Mapping Theorem* (often called *Banach Contraction Mapping Theorem*).
Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which there exists a constant c such that $0 < c < 1$, and

$$|f(x) - f(y)| \le c\,|x - y| \,, \qquad \forall x, y \in \mathbb{R} \,.$$

This can also be stated as saying that f is Lipschitz with Lipschitz constant < 1. Geometrically speaking, this means that the distance between the images $f(x)$ and $f(y)$ is no greater than c times the distance between the original points x and y.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence $(y_n)_{n\in\mathbb{N}}$ iteratively by setting

$$y_{n+1} = f(y_n) \,.$$

Show that (y_n) is a Cauchy sequence. This allows you to conclude that (y_n) converges; let $y = \lim y_n$.
Hint: Show that $|y_{m+1} - y_{m+2}| \le c^m |y_1 - y_2|$, then use the formula for geometric series to show that, for any $m < n$, $|y_m - y_n| \le \dfrac{c^{m-1}}{1-c}\,|y_1 - y_2|$, and use this to prove that (y_n) is Cauchy.

(c) Prove that y (defined in part (b)) is a fixed point of the function f, i.e., that

$$f(y) = y \,.$$

(d) Prove that y (defined in part (b)) is the unique fixed point of the function f. This implies, in particular, that for any $x \in \mathbb{R}$, the sequence of iterates $(x, f(x), f(f(x)), \ldots)$ converges to y.

Food for Thought: Davidson and Donsig, Exercises 9.1/J, 7.1/A, 7.1/C, 7.1/D