MATH 5163 Homework 9 Due Fri, 4/18/14, 5 p.m.

Problem 1. [Spaces of sequences]

Let ℓ^p, ℓ^∞, and s stand for linear spaces of sequences $\mathbf{x} = (x_j)_{j\in\mathbb{N}} := (x_1, x_2, \ldots)$ endowed with the norms

$$\|\mathbf{x}\|_{\ell^p} := \left(\sum_{j=1}^{\infty} |x_j|^p\right)^{1/p} \qquad \text{for } p \in [1,\infty)\ ,$$

$$\|\mathbf{x}\|_{\ell^\infty} := \sup_{j\in\mathbb{N}} |x_j|\ ,$$

$$\|\mathbf{x}\|_{s} := \sum_{j=1}^{\infty} \frac{1}{2^j}\,\frac{|x_j|}{1+|x_j|}\ .$$

(a) Let $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ be the sequence of infinite sequences $\mathbf{x}^{(n)}$, where

$$\mathbf{x}^{(n)} := \Big(\underbrace{1,1,\ldots,1}_{n\ \text{terms}},0,0,\ldots\Big)\ . \tag{1}$$

Show that the sequence $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ converges to the sequence $\mathbf{1} = (1,1,\ldots)$ in the space s.

(b) Demonstrate that the sequence $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ given by (1) is not a Cauchy sequence (and, hence, does not converge) in ℓ^∞.

(c) Does the sequence $(\mathbf{x}^{(n)})_{n\in\mathbb{N}}$ given by (1) converge in ℓ^p for some $p \in [1,\infty)$?

(d) Let $(\mathbf{y}^{(n)})_{n\in\mathbb{N}}$ be the sequence of infinite sequences $\mathbf{y}^{(n)}$, where

$$\mathbf{y}^{(n)} := \Big(\underbrace{\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}}_{n\ \text{terms}},0,0,\ldots\Big)\ . \tag{2}$$

Show that this sequence converges in s to the sequence $\mathbf{0} = (0,0,\ldots)$.

(e) Show that the sequence $(\mathbf{y}^{(n)})_{n\in\mathbb{N}}$ given by (2) converges in ℓ^∞.

(f) Show that the sequence $(\mathbf{y}^{(n)})_{n\in\mathbb{N}}$ given by (2) converges in ℓ^p for $p \in (1,\infty)$.

(g) Does the sequence $(\mathbf{y}^{(n)})_{n\in\mathbb{N}}$ given by (2) converge in ℓ^1?

(h) Let $(\mathbf{z}^{(n)})_{n\in\mathbb{N}}$ be the sequence of infinite sequences $\mathbf{z}^{(n)}$, where

$$\mathbf{z}^{(n)} := \Big(\underbrace{\frac{1}{n^\alpha},\frac{1}{n^\alpha},\ldots,\frac{1}{n^\alpha}}_{n\ \text{terms}},0,0,\ldots\Big)\ .$$

For which $p \in (0,\infty)$ does this sequence converge in ℓ^p?

Problem 2. [Equivalence of norms]

Two norms, $\|\cdot\|$ and $\|\cdot\|'$, on the same linear space V are said to be *equivalent* if there exist positive constants C_1 and C_2 such that

$$C_1\|\mathbf{x}\| \le \|\mathbf{x}\|' \le C_2\|\mathbf{x}\| \qquad \forall\, \mathbf{x} \in V \ .$$

In this problem let $\mathbb{R}^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R}\}$ stand for the linear space of finite sequences of n numbers, $\mathbb{R}^\infty = \{\mathbf{x} = (x_1, x_2, \ldots) : x_j \in \mathbb{R}\}$ stand for the linear space of infinite sequences, and let $\|\cdot\|_p$ denote the ℓ^p-norm in $\mathbb{R}^n$ or $\mathbb{R}^\infty$:

$$\|\mathbf{x}\|_p := \left(\sum_j |x_j|^p\right)^{1/p} \quad \text{for } p \in [1,\infty) \ ; \qquad \|\mathbf{x}\|_\infty := \sup_j |x_j| \ .$$

(a) Prove that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$ are equivalent.

(b) Prove that the norms $\|\cdot\|_2$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$ are equivalent.

(c) Use your results from parts (a) and (b) to show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathbb{R}^n$ are equivalent.

(d) Give an example to show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent norms on the space $\mathbb{R}^\infty := \{\mathbf{x} = (x_1, x_2, \ldots) : x_j \in \mathbb{R}\}$.

(e) Give an example to show that $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent norms on the space $\mathbb{R}^\infty$ from part (d).

Problem 3. [Comparison of L^p for different p; Hölder inequality]

Let Ω be an open subset of $\mathbb{R}^n$, $L^p(\Omega)$ be the linear space of functions $f : \Omega \to \mathbb{R}$ with $\|f\|_{L^p(\Omega)} < \infty$, and in all parts of this problem assume that

$$1 \le p_1 < p_2 \le \infty \ .$$

(a) Use the Hölder inequality,

$$\left|\int_\Omega f(x)\, g(x)\, \mathrm{d}x\right| \le \|f\|_{L^p(\Omega)}\, \|g\|_{L^q(\Omega)}$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$ if $p, q \in (1,\infty)$, or $p = 1$ and $q = \infty$, to show that

$$\left|\int_\Omega |h(x)|^{p_1}\, \mathrm{d}x\right| \le |\Omega|^{1-p_1/p_2}\, \|h\|^{p_1}_{L^{p_2}(\Omega)} \ ,$$

where $|\Omega| := \displaystyle\int_\Omega 1\, \mathrm{d}x$ is the volume of the domain Ω. Please write your derivation in detail.

Hint: Set $p = p_2/p_1 > 1$, $f = |h|^{p_1}$, and choose g appropriately.

(b) Use your result for part (a) to show that $L^{p_2}(\Omega) \subset L^{p_1}(\Omega)$ if $|\Omega| < \infty$.

(c) Give a simple example to show that the inclusion in part (b) is strict, i.e., for $|\Omega| < \infty$, find a function $f \in L^{p_1}(\Omega)$ such that $f \notin L^{p_2}(\Omega)$.

Hint: Consider $f(x) = \dfrac{1}{x^\alpha}$ for $x \in (0,1)$ where $\alpha > 0$ is appropriately chosen.

(d) Give a simple example that shows that the inclusion from part (b) is not true if $|\Omega|$ is infinite.

Hint: How about $f(x) = \dfrac{1}{x^\alpha}$ for $x \in (1,\infty)$ for an appropriate choice of $\alpha > 0$?

Problem 4. [Continuity of operators]

Let $C[0,1]$ be the linear space of continuous functions on $[0,1]$ with the norm

$$\|f\|_{C[0,1]} := \sup_{x\in[0,1]} |f(x)| \ ,$$

and $C^1[0,1]$ be the linear space of C^1 functions on $[0,1]$ with the norm

$$\|f\|_{C^1[0,1]} := \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} |f'(x)| \ ,$$

(a) Let $A : C[0,1] \to C[0,1]$ be the linear integral operator

$$(Af)(x) := \int_0^x f(y)\,\mathrm{d}y \ .$$

Show that the operator A is continuous.

(b) Let the linear integral operators B be defined by

$$(Bf)(x) := \int_0^1 K(x,y)\,f(y)\,\mathrm{d}y \ ,$$

where $K \in C^1([0,1]\times[0,1])$. Show that B is a continuous map from $C[0,1]$ to $C^1[0,1]$.

(c) Prove that the nonlinear operator $E : C[0,1] \to C[0,1]$ defined by

$$(Ef)(x) := f(x)^2$$

is continuous.