MATH 5453 Homework 9 Due Thu, Nov 6

Problem 33 from Section 2.4 of the book.

Additional question: Construct a function sequence satisfying the conditions of Problem 2.4/33 for which $\int f \neq \liminf \int f_n$.

Hint to Problem 2.4/33: This is very easy if you use Theorem 2.30 and some of the famous convergence theorems.

Additional problem 1.

Let $\{f_n\}$ and $\{g_n\}$ be real-valued function sequences.

(a) Prove that if $f_n \to f$ in measure, then $\alpha f_n \to \alpha f$ in measure for any $c \in \mathbb{R}$.

(b) Prove that if $f_n \to f$ in measure and $g_n \to g$ in measure, then $f_n + g_n \to f + g$ in measure.

Hint to (b): The triangle inequality implies that

$$\{x : |f_n(x) + g_n(x) - f(x) - g(x)| < \epsilon\} \supset \left\{x : |f_n(x) - f(x)| < \frac{\epsilon}{2}\right\} \cap \left\{x : |g_n(x) - g(x)| < \frac{\epsilon}{2}\right\} .$$

Take the complement of this inclusion, then take the measure of both sides, use DeMorgan's laws and Boole's inequality, and take the limit $n \to \infty$.

Additional problem 2.

Consider the function sequence $\{f_n\}_{n=1}^{\infty}$, where $f_n : [0, \infty) \to \mathbb{R}$ is defined by $f_n(x) = \frac{x}{n}$, and let μ be the Lebesque measure on $[0, \infty)$. Use the sequence $\{f_n\}_{n=1}^{\infty}$ to show that the condition $\mu(X) < \infty$ in Egoroff's Theorem is indeed necessary.

Additional problem 3.

(a) Show that the condition $\lim\limits_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > 0\}) = 0$ implies that $f_n \to f$ in measure on X.

(b) Give an example of a sequence of functions $f_n : [0, 1] \to \mathbb{R}$ converging in measure to a function $f : [0, 1] \to \mathbb{R}$ but such that $\lim_{n\to\infty} \mu(\{x \in X : |f_n(x) - f(x)| > 0\}) \neq 0$. This shows that the converse of (a) is false.

(c) Show that the condition in (a) implies that for μ-almost all $x \in X$, $f_n(x) = f(x)$ for infinitely many $n \in \mathbb{N}$. This is equivalent to showing that μ-almost all $x \in X$ belong to the set $\limsup\{x \in X : f_n(x) = f(x)\}$.

Additional problem 4.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Show that the sequence converges to a real number a if and only if every subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ has a subsequence $\{a_{n_{k_j}}\}_{j=1}^{\infty}$ converging to a.

Remark: This fact holds also for sequences $\{a_n\}_{n=1}^{\infty}$ for which $\lim_{n\to\infty} a_n = -\infty$ or for which $\lim_{n\to\infty} a_n = \infty$ (but you do not need to prove this here).

Additional problem 5.

Let μ be a Lebesgue-Stieltjes measure on $\mathbb{R}$, $\mathcal{M}$ be the domain of μ, and $L^1(\mu)$ be the linear space of real-valued integrable functions. (All statements below are true also for complex-valued functions.)

(a) Let $f \in L^1(\mu)$ and $\epsilon > 0$. Then there exists an integrable simple function $\phi = \sum_j a_j \chi_{E_j}$ (where, without loss of generality, we assume that all numbers a_j are non-zero) such that $\int |f - \phi| \, \mathrm{d}\mu < \epsilon$.

Hint: Let $\{\phi_n\}_{n=1}^{\infty}$ be a sequence of simple functions as in Theorem 2.10, and apply the Dominated Convergence Theorem to the sequence $\{|\phi_n - f|\}_{n=1}^{\infty}$.

Remark: This statement means that the integrable simple functions are dense in the space $L^1(\mu)$ endowed with the $L^1(\mu)$-metric.

(b) Show that the sets E_j in the notations of (a) have finite measure.

Hint: Recall that, in the notations of (a), we assumed that $a_j \neq 0$ for all j.

(c) Let $E \in \mathcal{M}$, and $\mu(E) < \infty$. Then for every $\epsilon > 0$ there exists a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \epsilon$.

Hint: Use Theorem 1.18.

(d) Show that, if E and F are measurable sets, $\mu(E \triangle F) = \int |\chi_E - \chi_F| \, \mathrm{d}\mu$.

(e) Combine the previous results to show that each integrable function f can be approximated arbitrarily closely in the $L^1(\mu)$-metric by an integrable simple function $\phi = \sum_j b_j \chi_{A_j}$, where A_j are open intervals.

(f) Let (a, b) be an open interval. Show that we can approximate $\chi_{(a,b)}$ in the L^1 metric arbitrarily closely by continuous functions that vanish outside (a, b) (i.e., are equal to zero when their argument is not in (a, b)).

(g) Prove that the set of compactly supported continuous functions is dense in $L^1(\mu)$, i.e., that for any $\epsilon > 0$, there exists a continuous function $g \in L^1(\mu)$ that vanishes outside a bounded interval and such that $\int |f - g| \, \mathrm{d}\mu < \epsilon$.