MATH 5763 Homework 9 Due Fri, 4/22/11

Problem 1. Let $X = \{X_t : t \geq 0\}$ be a time-homogeneous continuous-time Markov process with state space $\mathcal{X} = \eta\mathbb{Z}$, where $\eta\mathbb{Z}$ is a shorthand notation for the set of all integer multiples of η:

$$\eta\mathbb{Z} := \{\ldots, -2\eta, -\eta, 0, \eta, 2\eta, \ldots\} .$$

The process X is allowed to jump "up" or "down" by η with equal probabilities (like in the case of a symmetric simple random walk). Let the intensity of the process X be τ, i.e.,

$$p_{jk}(h) := \mathbb{P}\big(X(t+h) = k\eta \,\big|\, X_t = j\eta\big) = \begin{cases} \tau h + o(h) & \text{for } k = j \pm 1 \ , \\ 1 - 2\tau h + o(h) & \text{for } k = j \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

(a) Show that the probabilities $p_k(t) := \mathbb{P}(X_t = k\eta)$ satisfy the system of ODEs

$$p_k'(t) = \tau \left[p_{k-1}(t) - 2p_k(t) + p_{k+1}(t)\right] .$$

(b) Use the system of ODEs for the probabilities $p_k(t)$ to show that the characteristic function

$$\phi(\xi, t) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\xi X_t}\right] = \sum_{k\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\xi k\eta}\, p_k(t)$$

satisfies the equation $\dfrac{\partial \phi}{\partial t} = \tau\left(\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta}\right)\phi$.

(c) Assume that at $t = 0$, the process was at 0 (i.e., $X_0 = 0$). What does this imply for $\phi(\xi, 0)$? Solve the equation for $\phi(\xi, t)$ derived in part (b) with the initial condition you just found.

Hint: Although the equation for $\phi(\xi, t)$ derived in (b) is about a function of two variables (namely, ξ and t), it does not contain ξ-derivatives, so you can solve it simply as an ordinary differential equation treating ξ as a fixed number. The ODE you then have to solve is of the simplest kind, $x'(t) = \alpha\, x(t)$, where $\alpha = \text{const}$.

(d) Now let the "spatial step-size" η of the process go to zero, and the "temporal intensity" τ of the process go to infinity, in such a way that $2\eta^2\tau \to 1$. Compare the expression for the characteristic function $\phi(\xi, t)$ in this limit with the characteristic function $\phi_{\mathrm{N}(\mu,\sigma^2)}(\xi) = \mathrm{e}^{\mathrm{i}\mu\xi - \frac{1}{2}\sigma^2\xi^2}$ of a normal random variable with mean μ and variance σ^2. What can you conclude about the distribution of the random variable X_t in the limit $\eta \to 0$, $\tau \to \infty$, $2\eta^2\tau \to 1$?

Hint: To perform the limiting transition, you can expand the expression $\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta}$ (which will be part of your result for $\phi(\xi, t)$) in a Taylor series with respect to η around the point $\eta = 0$, and, after the obvious cancellations, you will obtain

$$\mathrm{e}^{\mathrm{i}\xi\eta} - 2 + \mathrm{e}^{-\mathrm{i}\xi\eta} = -\eta^2\xi^2 + o(\xi^2) .$$

Problem 2. This problem is about distributions (generalized functions). In class we showed directly that $H' = \delta$, where $H = \chi_{[0,\infty)}$ is the Heaviside ("unit step") function. Here you will explore this relations by using Laplace transform, and will also obtain results about approximating Dirac delta-function by "rectangle" functions.

Throughout this problem, let a stand for a strictly positive number. Let $H_a := \chi_{[a,\infty)}$; treat this a a function defined on $\mathbb{R}$ or on $[0,\infty)$, depending on the context. Define the distribution δ_a by

$$\int_{\mathbb{R}} \delta_a(x)\,\psi(x)\,\mathrm{d}x = \psi(a) \ ,$$

where ψ is an arbitrary test function (i.e., an infinitely differentiable function with compact support).

(a) Prove directly that $H_a' = \delta_a$ in the sense that $\displaystyle\int_{\mathbb{R}} H_a'(x)\,\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \delta_a(x)\,\psi(x)\,\mathrm{d}x$.

Hint: Adapt the calculation we did in class.

(b) Directly from the definition of the Laplace transform compute the Laplace transforms $\mathcal{L}[H_a](\xi)$ and $\mathcal{L}[\delta_a](\xi)$ of H_a and δ_a. (Remember the assumption $a > 0$.)

(c) Directly from the definition of the Laplace transform prove that, for an arbitrary differentiable function $f : [0,\infty) \to \mathbb{R}$, $\mathcal{L}[f'](\xi) = \xi\mathcal{L}[f](\xi) - f(0)$.

(d) Apply the identity from part (c) formally to H_a to give a different proof to the result proved in part (a).

(e) For $\epsilon > 0$, let the function $g_{a,\epsilon} : [0,\infty) \to \mathbb{R}$ be defined as $g_{a,\epsilon} := \frac{1}{\epsilon}\,\chi_{[a,a+\epsilon]}$. Compute $\mathcal{L}[g_{a,\epsilon}]$ and find $\lim\limits_{\epsilon\to 0} \mathcal{L}[g_{a,\epsilon}]$. What is the moral of your result?

(f) In class we defined the nth derivative $\delta_a^{(n)}$ of δ_a by $\displaystyle\int_{\mathbb{R}} \delta_a^{(n)}(x)\,\psi(x)\,\mathrm{d}x := (-1)^n \psi^{(n)}(a)$.

Use this definition to find $\mathcal{L}[\delta_a^{(n)}]$.

(g) Often the formal notation $\delta(x-a)$ is used instead of δ_a, the reason behind this being that by treating δ as an ordinary function and changing the variable x to $y = x - a$, we obtain for any test function ψ

$$\int_{\mathbb{R}} \delta(x-a)\,\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}} \delta(y)\,\psi(y+a)\,\mathrm{d}y = \psi(y+a)\big|_{y=0} = \psi(a) \ .$$

Use this to find the Laplace transform $\mathcal{L}[\delta_a(\cdot + \Delta x)](\xi)$ of the function $\delta_a(x + \Delta x)$, where Δx is a constant. What is $\mathcal{L}\left[\dfrac{\delta_a(\cdot + \Delta x) - \delta_a(\cdot)}{\Delta x}\right]$?

(h) Take the limit $\Delta x \to 0$ of $\mathcal{L}\left[\dfrac{\delta_a(\cdot + \Delta x) - \delta_a(\cdot)}{\Delta x}\right]$. Compare this with your result in part (f); discuss briefly.

Problem 3. Let $\{W(t)\}_{t\geq 0}$ be a standard Wiener process (for which $W(t) \sim N(0,t)$). Let $m > 0$, fix $t > 0$, and consider the event $\{W(t) > m\}$. Since $W(s)$ is a continuous function for $0 \leq s \leq t$, and $W(0) = 0$, the Intermediate Value Theorem implies that if the event $W(t) > m$ occurs, then we have $W(s) = m$ for at least one $s \in [0,t]$. It is natural to be interested in the moment when the Wiener process reaches position m for the first time, so define

$$T_m := \inf\{s > 0 : W(s) = m\} .$$

Set

$$R(s) := \begin{cases} W(s) & \text{for } s < T_m , \\ 2m - W(s) & \text{for } s \geq T_m , \end{cases}$$

which may be envisaged as reflecting the portion of the path after T_m with respect to the horizontal line $\{y = m\}$. This construction – reminiscent of the Reflection Principle from Problem 4 of Homework 4 and Problem 5 of Homework 5 – is represented in the figure below; note that by the definition of T_m, $R(t) \leq m$ when $W(t) \geq m$.

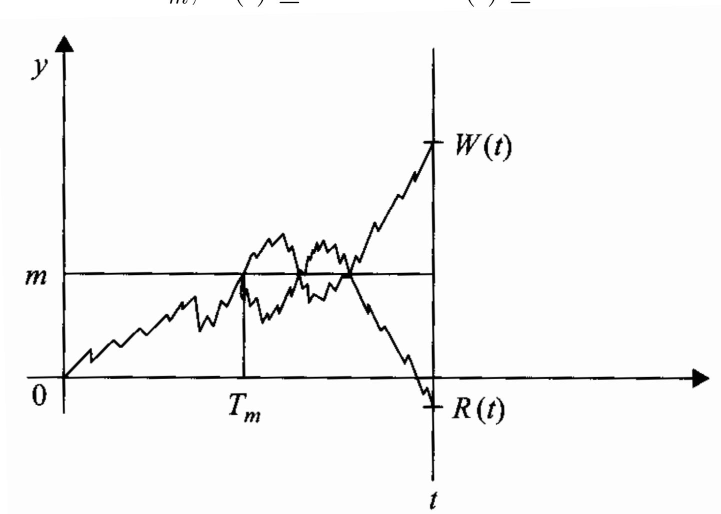

Figure 1: Sketch illustrating the reflection principle.

The argument of the Reflection Principle first asserts that the original and the reflected paths are equally likely, because of the symmetry of the normal distribution around its mean (here we are using the fact that $W(t) \sim N(0,t)$). Second, it observes that reflection is a one-to-one transformation, and finally it claims that we therefore have

$$\mathbb{P}(T_m \leq t,\, W(t) > m) = \mathbb{P}(T_m \leq t,\, W(t) < m) . \tag{1}$$

This observation yields a remarkably neat way to find the distribution of T_m, done in the steps below.

(a) How are the events $\{W(t) > m\}$ and $\{T_m \leq t\}$ related? Explain why this implies that

$$\mathbb{P}(T_m \leq t,\, W(t) > m) = \mathbb{P}(W(t) > m) . \tag{2}$$

(b) Note that the events $\{W(t) > m\}$ and $\{W(t) < m\}$ form a partition of the sample space. Use this fact together with the Reflection Principle (1) and the equality (2) to show that

$$\mathbb{P}(T_m \le t) = \mathbb{P}(|W(t)| > m) = \sqrt{\frac{2}{\pi t}} \int_m^\infty \mathrm{e}^{-x^2/(2t)}\, \mathrm{d}x = \int_0^t \frac{|m|}{\sqrt{2\pi y^3}}\, \mathrm{e}^{-m^2/(2y)}\, \mathrm{d}y \ .$$

Here we wrote absolute value of m in the final expression to make the formula applicable to any $m \in \mathbb{R}$, not just to positive values of m.

(c) What is the p.d.f. of the random variable T_m?

Food for thought. One can show that $\mathbb{P}(T_m < \infty) = 1$ by showing that $\int_0^\infty f_{T_m}(x)\, \mathrm{d}x = 1$. Think about the meaning of this fact. There is no need to do this calculation or to write anything about this in your write-up.

Problem 4. Consider a queue of type $\mathrm{M}(\lambda)/\mathrm{M}(\mu)/1$, i.e., a one-server queue at which the customers arrive as a Poisson process of rate λ (with inter-arrival times of type $\mathrm{Exp}(\lambda)$), and the service times are independent $\mathrm{Exp}(\mu)$ random variables. Due to the memorylessness of the exponential random variables, this stochastic process is Markov. Let $\rho := \frac{\mu}{\lambda}$ stand for the traffic intensity.

Suppose that at a given moment there are $k \ge 1$ customers in the queue (including the one being served). Let Z stand for the number of customers completing service before the next arrival. Show that the p.m.f. of Z is

$$p_Z(j) = \begin{cases} \dfrac{1}{1+\rho}\left(\dfrac{\rho}{1+\rho}\right)^j & \text{for } 0 \le j \le k-1 \ , \\[2ex] \left(\dfrac{\rho}{1+\rho}\right)^k & \text{for } j = k \ . \end{cases}$$

Hint: Think simply, without going into queueing theory. If $Z = j \le k-1$, then there are exactly j people served before the next new customer arrives. This means that

$$S_1 + \cdots + S_r < A \le S_1 + \cdots + S_{r+1} \ ,$$

where S_i are the service times and A is the time until the arrival of the next new customer. What kind of random variables are A and $T_r := S_1 + \cdots + S_r$? The event $\{T_r < A \le T_{r+1}\}$ can be expressed in terms of the events $\{A \le T_{r+1}\}$ and $\{A \le T_r\}$ – how can this fact be used to find its probability?