Techniques for the Oscillated Pendulum and the Mathieu Equation

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Abstract

In this paper, the problem of an inverted pendulum with vertical oscillation of its pivot is treated. The equation of motion is developed, and the stability of the pendulum is studied, through both analytical and numerical means.

1 Introduction

A pendulum has two critical points in its swing: the lowest position and the highest. When placed in either of these positions with no rotational speed, the pendulum will remain motionless. What is radically different between these two positions is their stability. The lower position, one of the classic examples of an approximate harmonic oscillator, is completely stable. The upper position is unstable, falling with even the slightest change in the position or velocity of the pendulum.

Both may change stability, however, if the anchor of the pendulum is moved in a vertical, oscillatory motion. The lower position's instability at certain frequencies is a result of parametric resonance, treated very eloquently in Arnol'd's *Ordinary Differential Equations* [1]. (For more on parametric resonance, see "Parametric Resonance" by Butikov [2]) The upper position's stability at certain frequencies is the opposite, the oscillation nullifying instead of amplifying movement. This paper will focus on the latter of the two phenomena.

This paper starts the examination of the stability of a hanging pendulum first by deriving and simplifying the differential equation that represents the motion of the pendulum, using the summations of forces and torques. Then, definitions and explanations of certain key tools, which are put into use further on, are given. These tools include stability and its characteristics, linear systems and matrices, and Floquet Theory.

The original problem is examined, first analytically, and then numerically. The analytical section includes the key approximations of the true, nonlinear equation, linearization and square wave approximations (the first turning the nonlinear equation into the so called Mathieu equation.) The condition for stability for the square wave approximation is found easily by using Floquet Theory. The condition for stability for the linearized equation, the Mathieu equation (as well as for a broader class of equations, Hill's Equation,) is then derived, also using Floquet Theory. Also included is a brief look at the corresponding damped equations. The numerics section includes comparisons between the three equations and the form that their stability/instability takes. Finally, approximations of an analytical solution to stability for the linearized equation and for the damped linear equation, are given.

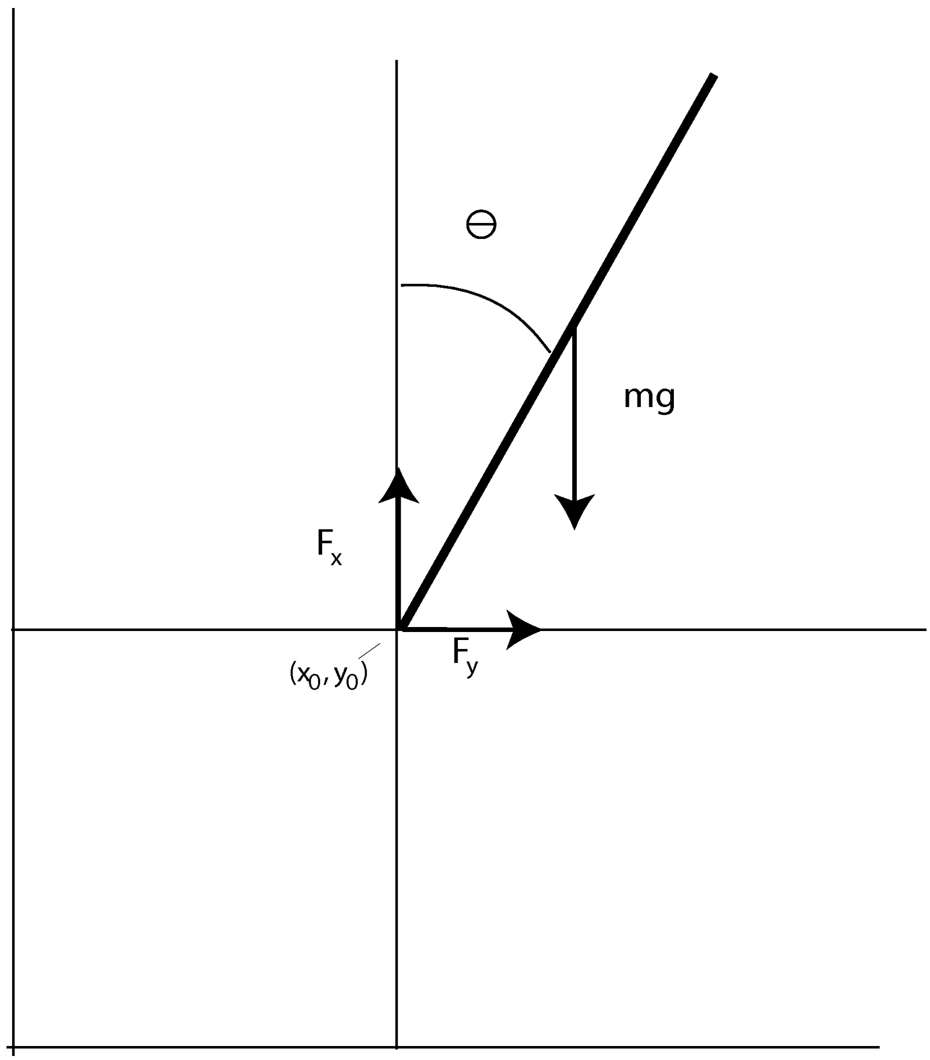


Figure 1: Inverted Pendulum

2 Derivation of the Equation of Motion

Assuming that the pendulum is a straight, thin, stiff rod, the forces are F_x in the x direction and F_y in the y direction on the lower end of the rod, and the force of gravity, mg on the center of the rod. (x_0, y_0) are the coordinates of the pivot. For brevity, let $\frac{\mathrm{d}w}{\mathrm{d}t} = \dot{w}$ and $\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} = \ddot{w}$.

$$y = y_0 + l\cos\theta \qquad x = x_0 + l\sin\theta$$

$$\ddot{y} = \ddot{y}_0 - l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta \qquad \ddot{x} = \ddot{x}_0 - l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta$$

Newton's Second Law and torque equations:

$$m\ddot{x} = F_x \qquad m\ddot{y} = F_y \qquad I_c\ddot{\theta} = F_y l\sin\theta - F_x l\cos\theta$$

Where I_c is the moment of inertia of the rod about the center. Solving and substituting for F_x and F_y in the torque equations:

$$I_c\ddot{\theta} = ml(\ddot{y} + g)\sin\theta - ml\ddot{x}\cos\theta$$

$$\frac{I}{ml}\ddot{\theta} = (\ddot{y}_0 + g)\sin\theta - \ddot{x}_0\cos\theta$$

Where $I_c + ml^2 = I$ is the general moment of inertia for any complex pendulum, so the pendulum need not be a straight, thin, stiff rod, as was assumed. Now, if the oscillation of the anchor is a vertical harmonic motion, then

$$x_0 = x_i \qquad y_0 = y_i + A\cos\omega t$$

$$\ddot{x}_0 = 0 \qquad \ddot{y}_0 = -A\omega^2\cos\omega t$$

$$\ddot{\theta} = \frac{ml}{I}(g - A\omega^2\cos\omega t)\sin\theta \tag{1}$$

This is the general equation of motion. However, it is useful to introduce dimensionless variables instead of m, I, g, etc. So instead, $\alpha = \frac{mlA}{I}$, $\beta = \frac{mlg}{I\omega^2}$, and $\tau = \omega t$ will be used. α is the nondimensionalized amplitude of the motion of the pivot, β is the nondimensionalized gravity acceleration, and τ is the nondimensionalized time unit.

$$\frac{1}{\omega^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \left(\frac{mlg}{I\omega^2} - \frac{mlA}{I}\cos\omega t\right)\sin\theta$$

becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} = (\beta - \alpha\cos\tau)\sin\theta$$

Once again, $\theta'' = \frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}$, so that

$$\theta'' = (\beta - \alpha\cos\tau)\sin\theta \tag{2}$$

In order to find the stability conditions for this equation, the tools of stability are needed.

3 Stability

3.1 Definitions of Stability

Consider the homogeneous, autonomous differential equation

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^2 \tag{3}$$

Assume that (3) has an equilibrium position and choose the coordinates x_i so that the equilibrium position is the origin, $\mathbf{v}(\mathbf{0}) = \mathbf{0}$

Definition 1 *The equilibrium position* $\mathbf{x} = \mathbf{0}$ *is called* ***stable*** *(or* ***Lyapunov stable****) if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *(depending only on* ϵ *and not on* t*) such that for every* $\mathbf{x_0}$ *for which* $||\mathbf{x_0}|| < \delta$ *the solution* $\boldsymbol{\psi}(t)$ *of (3) with the initial condition* $\boldsymbol{\psi}(0) = \mathbf{x_0}$ *satisfies the inequality* $||\boldsymbol{\psi}(t)|| < \epsilon$ *for all* $t > 0$*.*

Definition 2 *The equilibrium position is called* ***asymptotically stable*** *if it is (Lyapunov) stable and*

$$\lim_{t\to\infty} \boldsymbol{\psi}(t) = \mathbf{0}$$

for every solution $\boldsymbol{\psi}(t)$ *with an initial condition lying within a sufficiently small neighborhood of* $\mathbf{0}$*.*

Definition 3 *The equilibrium position is called* ***unstable*** *if it is not stable.*

(Definitions directly from Arnol'd's *Ordinary differential Equations* [1].)

By these definitions, stable positions are asymptotically stable for all linear, homogeneous, ordinary differential equations (or ODEs) except for neutral centers, physically, harmonic oscillators with no friction.

3.2 Linear Systems

The easiest way to determine stability for second order, linear, homogeneous, *autonomous* ODEs is to examine the determinant and trace of their matrix, and then to fit them into the Trace-Determinant Plane.

$$x'' = ax + bx' \qquad \Longrightarrow \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Any $n \times n$ matrix $\mathbf{A}$ of a system of linear, homogeneous, autonomous, differential equations has n complex eigenvalues, $\lambda_1, \lambda_2, \ldots \lambda_n$ corresponding to linearly independant solutions

$$\psi_1(t) = C_1 e^{\lambda_1 t}, \psi_2(t) = C_2 e^{\lambda_2 t}, \ldots \psi_n(t) = C_n e^{\lambda_n t}$$

These eigenvalues are such that

$$\det \mathbf{A} = \lambda_1 \lambda_2 \ldots \lambda_n, \quad \operatorname{tr} \mathbf{A} = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

For real differential equations, the trace and determinant must be real. Note that the solutions, ψ_i are such that

$$\mathrm{Re}\lambda_i > 0 \Rightarrow \psi_i \to \infty, \quad \mathrm{Re}\lambda_i < 0 \Rightarrow \psi_i \to 0, \quad \mathrm{Re}\lambda_i = 0 \Rightarrow \psi_i \to C_i$$

If one real part of an eigenvalue is greater than zero, then the system is unstable. If all real parts are less than or equal to zero, then the system is Lyapunov stable. If all real parts are less than zero, then the system is asymptotically stable. For n=2, the trace and determinant completely determine the eigenvalues, and so determine the stability of the differential equations. If det < 0, then the eigenvalues are real and have different signs, and one or the other is greater than zero, so the system is unstable. If det > 0 and tr > 0, then the real parts of the eigenvalues have the same sign and both are positive, so the matrix is unstable. If det > 0 and tr < 0, then the real parts of the eigenvalues are both negative, and the matrix is stable. The eigenvalues for all cases are given by

$$\lambda_i = \frac{1}{2}(\mathrm{tr}\,\mathbf{A} \pm \sqrt{\mathrm{tr}\,(\mathbf{A})^2 - 4\det\mathbf{A}})$$

If the ODE's determinant is positive and its trace is less than or equal to zero, the ODE is stable. If its determinant is positive and its trace is negative, the ODE is asymptotically stable. Otherwise, it is unstable. A very simple, easy test of stability. This even applies to nonlinear ODEs, because within a certain neighborhood of an equilibrium point, a nonlinear equation usually acts like a linear equation. (This linear equation is found through the *linearization* of the nonlinear equation.) The only nonlinear equation for which this method does not work is the neutral center, with positive determinant and zero trace. In this case, the nonlinearity must be taken into account.

Unfortunately, this method does not apply to the problem at hand, even after linearization, because (2) is nonhomogeneous. Since the ODE depends on time, its trace and determinant are *changing*. The behaviors predicted by the Trace-Determinant Plane will hold for small changes in time, but in general, the behavior of the equation will be vastly different, due to crossing boundaries in the Plane or parametric resonance.

There is a way, however, to analyze the stability of periodic, nonhomogeneous ODEs, as this one is.

3.3 Floquet Theory

Consider the nonhomogeneous, periodic ODE

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}(t, \mathbf{x}), \quad \mathbf{v}(t+T, \mathbf{x}) = \mathbf{v}(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2 \tag{4}$$

with solution $\mathbf{x} = \boldsymbol{\psi}(t)$.

The solution $\boldsymbol{\psi}(t_2)$ can be related to $\boldsymbol{\psi}(t_1)$ by a map such that

$$\boldsymbol{\psi}(t_2) = \mathbf{A}_{t_1}^{t_2}\boldsymbol{\psi}(t_1)$$

This map is linked directly to $\mathbf{v}(t, \mathbf{x})$, so that it, too, is not affected by period changes: $\mathbf{A}_{t_1+nT}^{t_2+nT} = \mathbf{A}_{t_1}^{t_2}$ because $\mathbf{v}(t, \mathbf{x})$ is periodic in T. Of special importance is the transformation over one period, $\mathbf{A}_t^{t+T}$. This will be denoted by $\mathbf{G} = \mathbf{A}_t^{t+T}$.

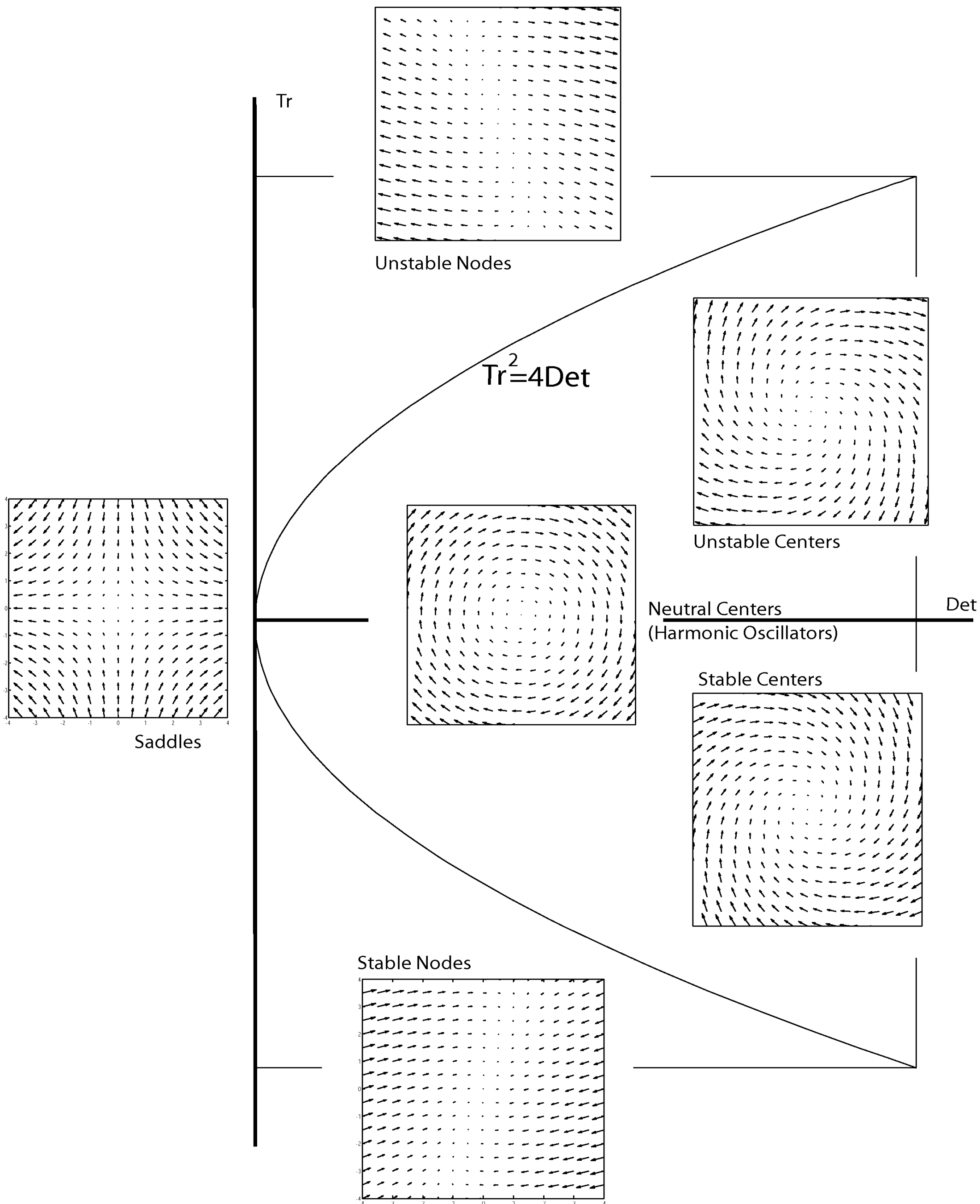


Figure 2: Trace Determinant Plane

Note that $\mathbf{A}_t^{t+nT} = \mathbf{G}^n$, since

$$\begin{aligned}
\mathbf{A}_t^{nT+t} \cdot \boldsymbol{\psi}(t) = \boldsymbol{\psi}(nT+t) &= \mathbf{A}_{(n-1)T+t}^{nT+t} \cdot \boldsymbol{\psi}((n-1)T+t) \\
&= \mathbf{A}_{(n-1)T+t}^{nT+t} \cdot \mathbf{A}_{(n-2)T+t}^{(n-1)T+t} \cdot \ldots \cdot \mathbf{A}_{T+t}^{2T+t} \cdot \mathbf{A}_t^{T+t} \cdot \boldsymbol{\psi}(t) \\
&= (\mathbf{A}_t^{T+t})^n \cdot \boldsymbol{\psi}(t) = \mathbf{G}^n \cdot \boldsymbol{\psi}(t)
\end{aligned}$$

so

$$\mathbf{A}_t^{t+nT} = \mathbf{G}^n$$

The importance of the mapping $\mathbf{G}$ is demonstrated in how properties of (4) correspond to properties of $\mathbf{G}$:

Theorem 1

1. *The point $\mathbf{x_0}$ is a fixed point of the mapping $\mathbf{G}$ (i.e., $\mathbf{G}\mathbf{x_0} = \mathbf{x_0}$) iff the solution $\boldsymbol{\psi}(t)$ with the initial condition $\boldsymbol{\psi}(0) = \mathbf{x_0}$ is periodic of period T.*
2. *A periodic solution $\boldsymbol{\psi}(t)$ is Lyapunov stable (or asymptotically stable) iff the fixed point $\mathbf{x_0}$ of $\mathbf{G}$ is Lyapunov stable (or asymptotically stable).*
3. *If the equation $\mathbf{x}' = \mathbf{v}(t, \mathbf{x})$ is linear, i.e. $\mathbf{v}(t, \mathbf{x}) = \mathbf{V}(t)\mathbf{x}$), then $\mathbf{G}$ is linear.*
4. *If the trace of $\mathbf{V}(t)$ is zero, then $\mathbf{G}$ preserves volume:* $\det(\mathbf{G}) = 1$.

Assertion four follows from Louville's Theorem:

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \int_{D(t)} \operatorname{div} \mathbf{v} \, \mathrm{d}x = \int_{D(t)} \operatorname{tr} \mathbf{G} \, \mathrm{d}x = 0 \Rightarrow V(t) = C$$

where $D(t)$ is the region under the action of the phase flow and $V(t)$ is the volume of the region.

Relating these maps to stability is Floquet Theory. As t approaches infinity, the solution of an unstable ODE generally approaches infinity, while every solution of a stable ODE remains finite. This relates directly to the map $\mathbf{G}$.

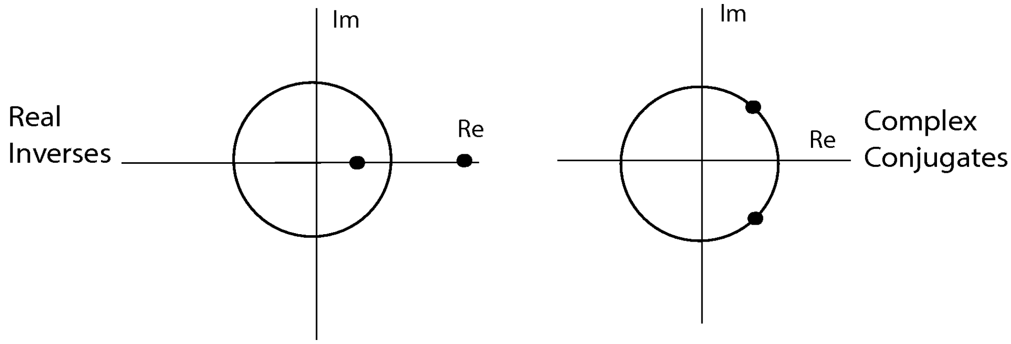
Unstable:

$$|\boldsymbol{\psi}(t)| \to \infty \Rightarrow |\mathbf{A}_0^n \cdot \boldsymbol{\psi}(0)| \to \infty \Rightarrow |\mathbf{G}^n \cdot \boldsymbol{\psi}(0)| \to \infty \Rightarrow |\lambda_1^n| \to \infty \text{ or } |\lambda_2^n| \to \infty$$

Stable:

$$|\boldsymbol{\psi}(\infty)| \nrightarrow \infty \Rightarrow \ldots\ldots |\lambda_1^n| \nrightarrow \infty \text{ and } |\lambda_2^n| \nrightarrow \infty$$

where λ_1 and λ_2 are the eigenvalues of $\mathbf{G}$. Since $\lambda_1\lambda_2 = \det \mathbf{G} = 1$, either λ_1 and λ_2 are real inverses of each other, i.e. $\lambda_1 = \frac{1}{\lambda_2}$, or they are complex conjugates where $|\lambda_1| = |\lambda_2| = 1$. If they are real inverses, then either $|\lambda_1|$ or $|\lambda_2|$ is larger than 1 (except for the case $\lambda_1 = \lambda_2 = \pm 1$, trivial), and so the mapping $\mathbf{G}$ and the equilibrium point of the original ODE are unstable. If they are complex conjugates, then $|\lambda_1| = |\lambda_2| = 1$, and the mapping $\mathbf{G}$ and the equilibrium point are Lyapunov stable. So the only information that is needed to determine stability is whether the eigenvalues are real inverses or complex conjugates. There is an extremely easy method for determining this.



Theorem 2 *Let* $\mathbf{G}$ *be the matrix of an area preserving linear transformation, i.e.,* $\det \mathbf{G} = 1$. *Then the mapping* $\mathbf{G}$ *is Lyapunov stable if the absolute value of the trace of that mapping is less than two,* $|\operatorname{tr} \mathbf{G}| < 2$*, and is unstable if the absolute value of the trace is greater than two,* $|\operatorname{tr} \mathbf{G}| > 2$.

Real eigenvalues:

$$|\operatorname{tr} \mathbf{G}| = |\lambda_1 + \lambda_2| = \left| x + \frac{1}{x} \right| > 2$$

Complex eigenvalues:

$$|\operatorname{tr} \mathbf{G}| = |\lambda_1 + \lambda_2| = |x + iy + x - iy| = |2x|, \ 1 = \sqrt{x^2 + y^2} \Rightarrow |x| < 1 \Rightarrow |\operatorname{tr} \mathbf{G}| < 2$$

This is the method that will be used in the forthcoming sections. (For more on stability, see Arnol'd Section 23 [1] or Verhulst [3].)

4 Analytic Methods

4.1 Linearization

$$\theta'' = (\beta - \alpha \cos \tau) \sin \theta$$

This equation is short and exact, but is difficult, if not impossible, to solve analytically. Even the simple question of stability or instability around $\theta = 0$ at a certain α and β may be impossible to answer analytically. The equation is nonlinear, depending on $\sin \theta$, and, perhaps worse, is nonautonomous, depending on $\cos \tau$.

First and foremost, the nonlinearity will be eliminated by approximation. Linearization, while uncomfortable, is necessary, $\sin \theta \approx \theta$ near $\theta = 0$, thus arriving at

$$\theta'' = (\beta - \alpha \cos \tau)\theta \tag{5}$$

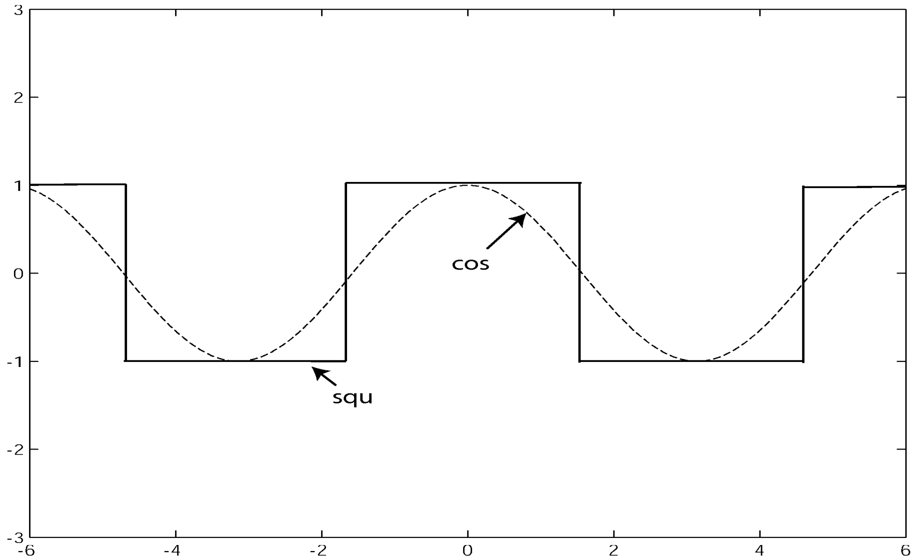


Figure 3: $\operatorname{squ}\tau$ vs. $\cos\tau$

This equation is known as the Mathieu equation, or at least one of its forms. Now, already, the results are only approximate, since this can only be done if θ does not stray too far from zero, as it does in some cases. The nonautonomous part of the equation is harder to deal with. There is one application of Floquet Theory that works well, and solves for stability/instability, but only for a different, even more approximate equation, the square wave equation. Thus the application will show the "form" of how the (5) acts, but will give different values of α and β for stability than the linearized equation would. It is correct in principle, but not in the actual numeric values. There is also a different application, which would solve exactly for the linearized equation's stability/instability, but requires an infinite summation of infinite products of hyperbolic sines and cosines. It is not useful until this difficulty is worked out.

4.2 Square Wave Approximation

It is the $\cos\tau$ in (5) that is causing trouble, so it will be exchanged for a function that acts somewhat like the cosine, but to which it is easier to apply Floquet Theory: the square wave. Thus (5) becomes

$$\theta'' = (\beta - \alpha \operatorname{squ}\tau)\theta \tag{6}$$

$$\text{where} \qquad \operatorname{squ}\tau = \begin{cases} 1, & 2n\pi - \frac{\pi}{2} < \tau \le 2n\pi + \frac{\pi}{2} \\ -1, & 2n\pi + \frac{\pi}{2} < \tau \le 2n\pi + \frac{3\pi}{2} \end{cases}, \qquad n \in \mathbb{Z}$$

This equation is easier to use for the problem because Floquet Theory becomes extremely unwieldy when dealing with linear ODEs for which the solution is not known. This square wave approximation *is* solvable. We write the ODE in matrix form:

$$\begin{pmatrix}\theta\\ \theta'\end{pmatrix}' = \begin{pmatrix}0 & 1\\ \beta-\alpha\operatorname{squ}\tau & 0\end{pmatrix}\begin{pmatrix}\theta\\ \theta'\end{pmatrix}$$

To find $\mathbf{A}_x^{x+T}$, it is beneficial to choose x such that it reduces the computation required. Without loss of generality, we can choose $x=-\frac{\pi}{2}$, $\mathbf{G}=\mathbf{A}_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}$. Also, finding $\mathbf{A}_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}$ is equivalent to finding $\mathbf{A}_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$ and $\mathbf{A}_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$ and multiplying them together.

$$\mathbf{A}_a^c\cdot\boldsymbol{\psi}(a)=\boldsymbol{\psi}(c)=\mathbf{A}_b^c\cdot\boldsymbol{\psi}(b)=\mathbf{A}_b^c\cdot\mathbf{A}_a^b\cdot\boldsymbol{\psi}(a)$$

So

$$\begin{aligned}
\text{for }\ -\tfrac{\pi}{2}<\tau\le\tfrac{\pi}{2},\qquad &\begin{pmatrix}\theta(\tau-\frac{\pi}{2})\\ \theta'(\tau-\frac{\pi}{2})\end{pmatrix}=\mathbf{A}_{-\frac{\pi}{2}}^{\tau-\frac{\pi}{2}}\cdot\begin{pmatrix}\theta(-\frac{\pi}{2})\\ \theta'(-\frac{\pi}{2})\end{pmatrix}\\
&\qquad=\begin{pmatrix}\cosh\omega_1\tau & \frac{1}{\omega_1}\sinh\omega_1\tau\\ \omega_1\sinh\omega_1\tau & \cosh\omega_1\tau\end{pmatrix}\cdot\begin{pmatrix}\theta(-\frac{\pi}{2})\\ \theta'(-\frac{\pi}{2})\end{pmatrix}\\
\text{for }\ \tfrac{\pi}{2}<\tau\le\tfrac{3\pi}{2},\qquad &\begin{pmatrix}\theta(\tau+\frac{\pi}{2})\\ \theta'(\tau+\frac{\pi}{2})\end{pmatrix}=\mathbf{A}_{\frac{\pi}{2}}^{\tau+\frac{\pi}{2}}\cdot\begin{pmatrix}\theta(\frac{\pi}{2})\\ \theta'(\frac{\pi}{2})\end{pmatrix}\\
&\qquad=\begin{pmatrix}\cosh\omega_2\tau & \frac{1}{\omega_2}\sinh\omega_2\tau\\ \omega_2\sinh\omega_2\tau & \cosh\omega_2\tau\end{pmatrix}\cdot\begin{pmatrix}\theta(\frac{\pi}{2})\\ \theta'(\frac{\pi}{2})\end{pmatrix}
\end{aligned}$$

where $\omega_1=\sqrt{\beta-\alpha}$, $\omega_2=\sqrt{\alpha+\beta}$. Therefore

$$\mathbf{A}_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=\begin{pmatrix}\cosh\pi\omega_1 & \frac{1}{\omega_1}\sinh\pi\omega_1\\ \omega_1\sinh\pi\omega_1 & \cosh\pi\omega_1\end{pmatrix},\quad \mathbf{A}_{\frac{\pi}{2}}^{\frac{3\pi}{2}}=\begin{pmatrix}\cosh\pi\omega_2 & \frac{1}{\omega_2}\sinh\pi\omega_2\\ \omega_2\sinh\pi\omega_2 & \cosh\pi\omega_2\end{pmatrix}$$

$$\mathbf{A}_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}=\begin{pmatrix}c_1c_2+\frac{\omega_1}{\omega_2}s_1s_2 & \frac{1}{\omega_1}s_1c_2+\frac{1}{\omega_2}c_1s_2\\ \omega_1s_1c_2+\omega_2c_1s_2 & c_1c_2+\frac{\omega_2}{\omega_1}s_1s_2\end{pmatrix}$$

where

$$c_i=\cosh\pi\omega_i,\quad s_i=\sinh\pi\omega_i$$

$$\left|\operatorname{tr}\mathbf{A}_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\right|=\left|2c_1c_2+\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\right)s_1s_2\right| \tag{7}$$

According to Theorem 2, if this quantity is smaller than two, then (6) is stable. If it is greater than two, then (6) is unstable.

It might be thought that this quantity is *never* less than two, but this is the case only if $\beta>\alpha$. If, however, $\alpha>\beta$, then $c_1=\cos\pi\sqrt{\alpha-\beta}$, $s_1=i\sin\pi\sqrt{\alpha-\beta}$, and $\omega_1=i\sqrt{\alpha-\beta}$, and (7) becomes:

$$\left|\operatorname{tr}\mathbf{A}_{-\frac{\pi}{2}}^{\frac{3\pi}{2}}\right|=\left|2c_1c_2+\left(\frac{\omega_2}{\omega_1}-\frac{\omega_1}{\omega_2}\right)s_1s_2\right| \tag{8}$$

where

$$
\begin{aligned}
c_1 &= \cos \pi\omega_1 & c_2 &= \cosh \pi\omega_2 \\
s_1 &= \sin \pi\omega_1 & s_2 &= \sinh \pi\omega_2 \\
\omega_1 &= \sqrt{\alpha - \beta} & \omega_2 &= \sqrt{\alpha + \beta}
\end{aligned}
$$

For a choice of α and β such that $\alpha > \beta$, the quantity *is* sometimes less than two. Note that the cases $\alpha < 0$ and $\beta < 0$ have not been discussed, though (7) still applies. The case $\alpha < 0$ is a reflection of $\alpha > 0$, since it can be thought of as a phase change of π in $\cos \tau$, which does not change the stability domains. The case $\beta < 0$ is not quite so simple, for it represents not an *inverted* pendulum undergoing vertical oscillation, but a *hanging* pendulum undergoing vertical oscillation. (7) and (8) work as well for this case as they do for $\beta > 0$, giving Figure 4.2, a graph of stability in the entire α-β plane. Stability is represented by white area and instability by black.

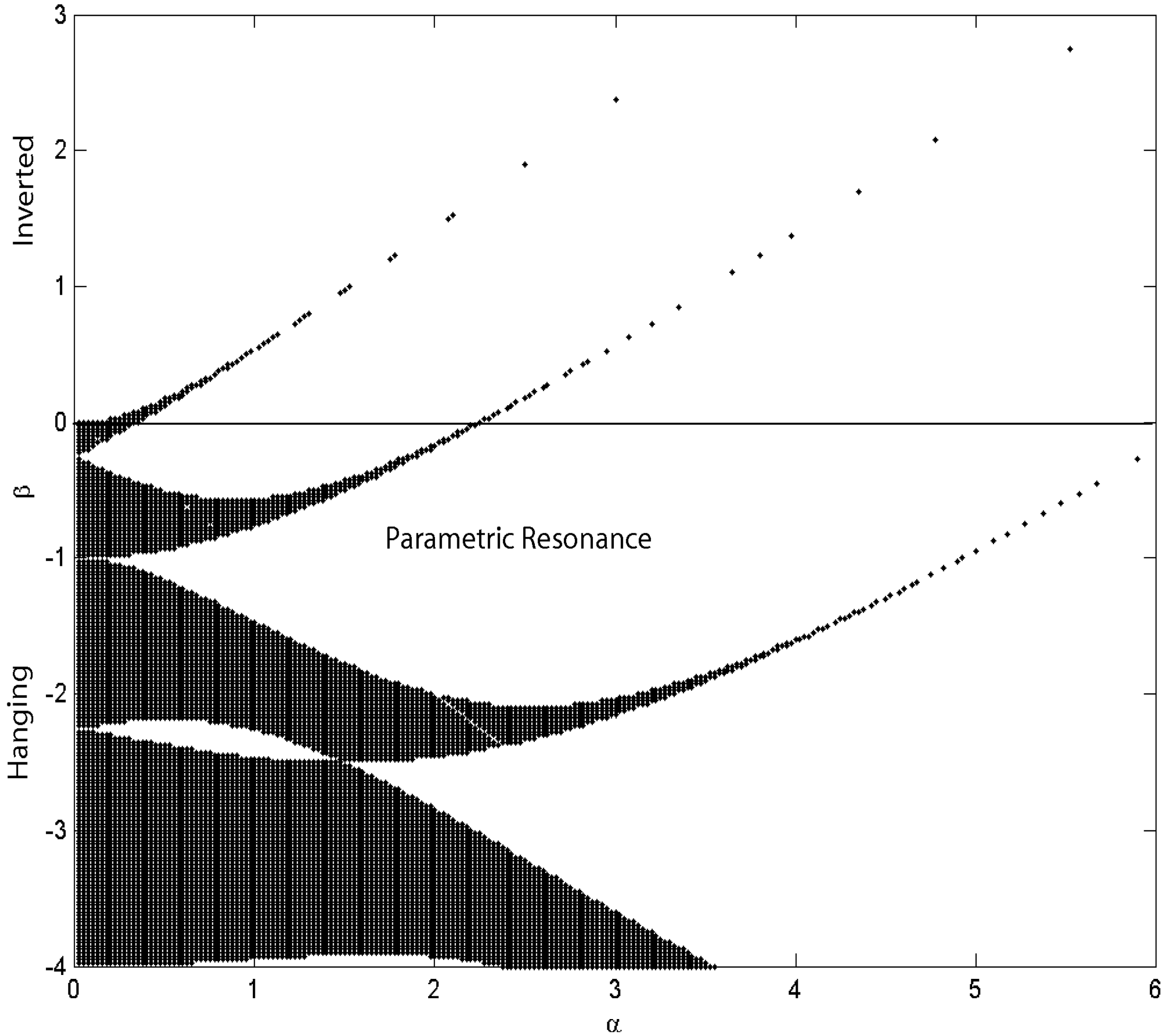


Figure 4: Stability and Parametric Resonance ($\alpha < 0$ trivial)

4.3 Floquet for a Linear, Frictionless ODE

The square wave approximation could be seen as an approximation of $\cos\tau$ by its values at two points. However, $\cos\tau$ is more accurately approximated by using more points. If approximated by a very large amount of points, $\cos\tau$ is modeled exactly. This is applied to find the solution matrix for (5) over one period, $\mathbf{A}_{\tau_0}^{\tau_0+T}$:

$$\mathbf{A}_{\tau_0}^{\tau_0+T} = \mathbf{A}_{\tau_0+T-\epsilon}^{\tau_0+T} \cdot \mathbf{A}_{\tau_0+T-2\epsilon}^{\tau_0+T-\epsilon} \cdot \ldots \cdot \mathbf{A}_{\tau_0+\epsilon}^{\tau_0+2\epsilon} \cdot \mathbf{A}_{\tau_0}^{\tau_0+\epsilon}$$

For very small changes of ϵ, the solution matrix is known:

$$\begin{pmatrix} \theta(\tau+\epsilon) \\ \theta'(\tau+\epsilon) \end{pmatrix} = \begin{pmatrix} \cosh(\omega\epsilon) & \frac{1}{\omega}\sinh(\omega\epsilon) \\ \omega\sinh(\omega\epsilon) & \cosh(\omega\epsilon) \end{pmatrix} \begin{pmatrix} \theta(\tau) \\ \theta'(\tau) \end{pmatrix} + O(\epsilon^2)$$

$$\mathbf{A}_{\tau}^{\tau+\epsilon} = \begin{pmatrix} \cosh(\omega\epsilon) & \frac{1}{\omega}\sinh(\omega\epsilon) \\ \omega\sinh(\omega\epsilon) & \cosh(\omega\epsilon) \end{pmatrix}, \tag{9}$$

where $\omega = \sqrt{\beta - \alpha\cos\tau}$

In order to find $\mathbf{A}_{\tau_0}^{\tau_0+T}$, an infinite number of these matrices, from $\tau = \tau_0$ to $\tau = \tau_0 + T$, must be multiplied together. The period T is once again 2π, and without loss of generality, τ_0 is chosen to be $-\pi$. Notations used:

$$k = \frac{2\pi}{\epsilon} = \text{number of “}\mathbf{A}\text{”s to be multiplied}$$

$$c_i = \cosh\omega_i\epsilon, \quad s_i = \sinh\omega_i\epsilon, \quad \omega_i = \sqrt{\beta - \alpha\cos(\tfrac{2\pi i}{k} - \pi)}$$

$$\mathbf{A}_{-\pi}^{\pi} = \begin{pmatrix} c_k & \frac{1}{\omega_k}s_k \\ \omega_k s_k & c_k \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} c_1 & \frac{1}{\omega_1}s_1 \\ \omega_1 s_1 & c_1 \end{pmatrix} = \prod_{i=1}^{k} \begin{pmatrix} c_i & \frac{1}{\omega_i}s_i \\ \omega_i s_i & c_i \end{pmatrix}$$

$$(\mathbf{A}_{-\pi}^{\pi})_{11} = c_1c_2\ldots c_{k-1}c_k + \frac{\omega_1}{\omega_2}s_1s_2c_3c_4\ldots + \frac{\omega_1}{\omega_3}s_1c_2s_3c_4\ldots + \ldots + \frac{\omega_1\omega_3}{\omega_2\omega_4}s_1s_2s_3s_4c_5\ldots + \ldots$$
$$(\mathbf{A}_{-\pi}^{\pi})_{22} = c_1c_2\ldots c_{k-1}c_k + \frac{\omega_2}{\omega_1}s_1s_2c_3c_4\ldots + \frac{\omega_3}{\omega_1}s_1c_2s_3c_4\ldots + \ldots + \frac{\omega_2\omega_4}{\omega_1\omega_3}s_1s_2s_3s_4c_5\ldots + \ldots$$

The top right and bottom left corners of the matrix do not matter. All that matters is the *trace* of the matrix:

$$\begin{aligned} \operatorname{tr}(\mathbf{A}_{-\pi}^{\pi}) = 2c_1c_2c_3\ldots &+ \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)s_1s_2c_3c_4\ldots + \left(\frac{\omega_1}{\omega_3} + \frac{\omega_3}{\omega_1}\right)s_1c_2s_3c_4\ldots + \ldots \\ \ldots &+ \left(\frac{\omega_1\omega_3}{\omega_2\omega_4} + \frac{\omega_2\omega_4}{\omega_1\omega_3}\right)s_1s_2s_3s_4c_5\ldots + \ldots\ldots \end{aligned} \tag{10}$$

This infinite summation of infinite products of hyperbolic sines and cosines is difficult, but there is some order to it. There are several properties that this summation exhibits:

1. If a term includes ω_i, then it also includes s_i; if it does not include ω_i, then it includes c_i.

2. There are $2n$ ωs in each term, $n \in \mathbb{N}$, n on the top of each ratio and n on the bottom of each ratio.

3. If ω_d and ω_e are on one side of a ratio, $d > e$, then there is at least one ω_f on the other side, such that $d > f > e$.

4. No ω_i is represented twice in the same ratio (no ω_i^2).

5. Every term within these rules is represented in the summation.

Also, (10) can be simplified for small ϵ, like in our case:

$$c_i c_j c_k \ldots \approx 1 \qquad \text{and} \qquad s_i \approx \omega_i \epsilon,$$

hence

$$\begin{aligned}\operatorname{tr}(\mathbf{A}^{\pi}_{-\pi}) = 2 &+ (\omega_1^2+\omega_2^2)\epsilon^2 + (\omega_1^2+\omega_3^2)\epsilon^2 + \ldots + (\omega_2^2+\omega_3^2)\epsilon^2 + \ldots \\ &\ldots + (\omega_1^2\omega_3^2+\omega_2^2\omega_4^2)\epsilon^4 + (\omega_1^2\omega_3^2+\omega_2^2\omega_5^2)\epsilon^4 + \ldots\ldots\end{aligned}$$

Each level of ϵ (ϵ^2, ϵ^4, etc.) contributes to the summation. A pattern must be found in order to integrate. Using property 3 of the summation and the Product Rule, this pattern is found.

On the order of ϵ^{2n} (n different ωs in each product)

$$P = \text{number of ways to have} \omega_{a_1}^2 \cdot \omega_{a_2}^2 \cdot \omega_{a_3}^2 \cdot \ldots \cdot \omega_{a_n}^2 :$$

$$= a_1 \cdot (a_2 - a_1) \cdot \ldots \cdot (a_n - a_{n-1}) + (a_2 - a_1) \cdot (a_3 - a_2) \cdot \ldots \cdot (a_n - a_{n-1}) \cdot (k - a_n)$$

$$= (k + a_1 - a_n) \cdot (a_2 - a_1) \cdot (a_3 - a_2) \cdot \ldots \cdot (a_n - a_{n-1})$$

So for the part of the trace on the order of ϵ^{2n}:

$$\sum_{a_1=1}^{k} \sum_{a_2=a_1}^{k} \cdots \sum_{a_{n-1}=a_{n-2}}^{k} \sum_{a_n=a_{n-1}}^{k} P\omega_{a_n}^2 \omega_{a_{n-1}}^2 \ldots \omega_{a_2}^2 \omega_{a_1}^2 \epsilon^{2n}$$

$$= \sum_{a_1=1}^{k} \sum_{a_2=a_1}^{k} \cdots \sum_{a_n=a_{n-1}}^{k} (k\epsilon + a_1\epsilon - a_n\epsilon)(a_n\epsilon - a_{n-1}\epsilon) \ldots (a_2\epsilon - a_1\epsilon)\omega_{a_n}^2 \omega_{a_{n-1}}^2 \ldots \omega_{a_2}^2 \omega_{a_1}^2 \epsilon^n$$

$$= \int_{-\pi}^{\pi}\int_{\tau_1}^{\pi}\cdots\int_{\tau_{n-1}}^{\pi} (2\pi + \tau_1 - \tau_n)(\tau_n - \tau_{n-1}) \ldots (\tau_2 - \tau_1)\omega_{\tau_n}^2 \omega_{\tau_{n-1}}^2 \ldots \omega_{\tau_2}^2 \omega_{\tau_1}^2 \; d\tau_n \; d\tau_{n-1} \ldots d\tau_2 \; d\tau_1$$

$$\operatorname{tr}(\mathbf{A}_{-\pi}^{\pi}) = 2 + \int_{-\pi}^{\pi} 2\pi\omega_{\tau_1}^2 \,\mathrm{d}\tau_1 + \int_{-\pi}^{\pi}\int_{\tau_1}^{\pi} \big(2\pi + \tau_1 - \tau_2)\big)(\tau_2 - \tau_1)\omega_{\tau_2}^2\omega_{\tau_1}^2 \,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 \tag{11}$$
$$+ \int_{-\pi}^{\pi}\int_{\tau_1}^{\pi}\int_{\tau_2}^{\pi} \big(2\pi + \tau_1 - \tau_3\big)(\tau_3 - \tau_2)(\tau_2 - \tau_1)\omega_{\tau_3}^2\omega_{\tau_2}^2\omega_{\tau_1}^2 \,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \dots$$

where

$$\omega_i = \sqrt{\beta - \alpha\cos i}$$

The remarkable thing about this formula is that at no point in its derivation is the fact that $\omega_\tau = \sqrt{\beta - \alpha\cos\tau}$ used. This means that the formula can be used not only for (5), Mathieu's Equation, but for a broader class of equations.

$$\theta'' = \omega(\tau)^2\theta, \tag{12}$$

where

$$\omega(\tau + T) = \omega(\tau)$$

This ODE is known as Hill's Equation. For any equation that falls into this category of ODEs, stability can be calculated using the derived formula.

$$\Bigg|2 + \int_{\tau_0}^{\tau_0+T} T\omega_{\tau_1}^2 \,\mathrm{d}\tau_1 + \int_{\tau_0}^{\tau_0+T}\int_{\tau_1}^{\tau_0+T} \big(T + \tau_1 - \tau_2\big)(\tau_2 - \tau_1)\omega_{\tau_2}^2\omega_{\tau_1}^2 \,\mathrm{d}\tau_2\,\mathrm{d}\tau_1$$
$$+ \int_{\tau_0}^{\tau_0+T}\int_{\tau_1}^{\tau_0+T}\int_{\tau_2}^{\tau_0+T} \big(T + \tau_1 - \tau_3\big)(\tau_3 - \tau_2)(\tau_2 - \tau_1)\omega_{\tau_3}^2\omega_{\tau_2}^2\omega_{\tau_1}^2 \,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \dots\Bigg| < 2 \tag{13}$$

Where τ_0 is any real number. So for any linear, frictionless, periodically nonautonomous second order differential equation, this summation of integrals determines stability or instability. This is a very important class of differential equations.

It was noted before that changing τ_0 does not change the value of the summation. This is not a property of the summation, but of each integral. It can be shown that changing τ_0 to τ_1 does not affect the value of the integrals, and so does not affect the value of the summation.

The derived integral summation can be generalized to higher order ODEs, though it loses much of its power for ODEs of order higher than two.

4.4 Higher Order ODEs

The purpose here is to develop an expression for the trace of the matrix from one period to the next of the following ODE:

$$\theta^{(n)} = f(\tau)\theta, \quad \text{where } f(\tau + T) = f(\tau) \tag{14}$$

Or in matrix form:

$$\begin{pmatrix} \theta \\ \theta' \\ \vdots \\ \theta^{(n-1)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ f(\tau) & 0 & \dots & \dots & 0 \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \\ \vdots \\ \theta^{(n-1)} \end{pmatrix}$$

First, the solution matrix over a small change in time is needed. As the solution matrix for second order ODEs was written in terms of hyperbolic cosines and hyperbolic sines, we will need a generalization of these. Define ω_τ such that $\omega_\tau^n = f(\tau)$. Now define

$$\mathrm{hyp}\,(n,j,x) = \sum_{t=0}^{n-1} \frac{1}{n}\, e^{-\frac{2\pi i t j}{n}}\, e^{(e^{\frac{2\pi i t}{n}})x}, \quad \text{where } n \in \mathbb{N}, j \in \mathbb{Z} \tag{15}$$

Several nice properties of this function are listed below.

1. $\mathrm{hyp}\,(1,0,x) = e^x$
2. $\mathrm{hyp}\,(2,0,x) = \cosh x$
3. $\mathrm{hyp}\,(2,1,x) = \sinh x$
4. $\mathrm{hyp}\,(n,j+n,x) = \mathrm{hyp}\,(n,j,x)$
5. $\sum_{j=0}^{n-1} \mathrm{hyp}\,(n,j,x) = e^x$
6. $\frac{\mathrm{d}}{\mathrm{d}x} \mathrm{hyp}\,(n,j,x) = \mathrm{hyp}\,(n,j-1,x)$

In addition, the Taylor Series about $x = 0$ for these functions (with $0 \le j < n$) is

$$\mathrm{hyp}\,(n,j,x) = \sum_{m=0}^{\infty} \frac{x^{mn+j}}{(mn+j)!}$$

e.g.

$$\mathrm{hyp}\,(n,0,x) = 1 + \frac{x^n}{n!} + \frac{x^{2n}}{(2n)!} + \frac{x^{3n}}{(3n)!} + \dots$$

The identity generalizing $\cosh^2 x - \sinh^2 x = 1$ is now developed:

$$1 = e^{(\sum_{m=0}^{n-1} e^{\frac{2\pi i m}{n}})x} = \prod_{m=0}^{n-1} e^{(e^{\frac{2\pi i m}{n}})x} = \prod_{m=0}^{n-1}\sum_{j=0}^{n-1} \mathrm{hyp}\,(n,j,e^{\frac{2\pi i m}{n}}x)$$

But

$$\begin{aligned}\mathrm{hyp}\,(n,j,e^{\frac{2\pi i m}{n}}x) &= \sum_{t=0}^{n-1} \frac{1}{n}\, e^{-\frac{2\pi i t j}{n}}\, e^{(e^{\frac{2\pi i t}{n}})\, e^{\frac{2\pi i m}{n}} x} = \sum_{t=1}^{n} \frac{1}{n}\, e^{-\frac{2\pi i (t-m) j}{n}}\, e^{(e^{\frac{2\pi i t}{n}})x} \\ &= e^{\frac{2\pi i m j}{n}}\, \mathrm{hyp}\,(n,j,x)\end{aligned}$$

So

$$\prod_{m=0}^{n-1}\sum_{j=0}^{n-1} e^{\frac{2\pi i m j}{n}}\, \mathrm{hyp}\,(n,j,x) = 1 \tag{16}$$

With these, we have the solution matrix. Label $\operatorname{hyp}(n, j, \omega_\tau \epsilon)$ with h_j.

$$\begin{pmatrix} \theta(\tau+\epsilon) \\ \theta'(\tau+\epsilon) \\ \vdots \\ \theta^{(n-1)}(\tau+\epsilon) \end{pmatrix} = \begin{pmatrix} h_0 & \frac{1}{\omega_\tau} h_1 & \dots & \frac{1}{\omega_\tau^{n-1}} h_{n-1} \\ \omega_\tau h_{n-1} & h_0 & \dots & \frac{1}{\omega_\tau^{n-2}} h_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_\tau^{n-1} h_1 & \omega_\tau^{n-2} h_2 & \dots & h_0 \end{pmatrix} \begin{pmatrix} \theta(\tau) \\ \theta'(\tau) \\ \vdots \\ \theta^{(n-1)}(\tau) \end{pmatrix}$$

Once again we are concerned with multiplying these matrices to acquire a full period map. The determinant of the map is 1, either by Louiville's Theorem or by noting that the determinant is exactly the identity acquired above. The trace of the matrix is what is now explored. Let $k = \frac{T}{\epsilon}$, let $\mathbf{A}_l = [a_{ij}]_l$ be the map from $\tau = \frac{Tl}{k}$ to $\frac{Tl}{k} + \epsilon$, and let $\mathbf{G}$ be the map from one period to the next. Then by definition of matrix multiplication,

$$\mathbf{G} = [g_{ij}] \mathbf{A}_n \mathbf{A}_{n-1} \dots \mathbf{A}_1 = [\sum_{1 \le x_{n-1}, \dots, x_1 \le n} a_{i x_{n-1}} a_{x_{n-1} x_{n-2}} \dots a_{x_1 j}]$$

$$\operatorname{tr} \mathbf{G} = \sum_{i=1}^{n} g_{ii} = \sum_{1 \le i, x_{n-1}, \dots, x_1 \le n} a_{i x_{n-1}} a_{x_{n-1} x_{n-2}} \dots a_{x_1 i} = \sum_{1 \le x_n, x_{n-1}, \dots, x_1 \le n} a_{x_n x_{n-1}} a_{x_{n-1} x_{n-2}} \dots a_{x_1 x_n}$$

And

$$a_{x_l x_{l-1}} = \omega_l^{x_l - x_{l-1}} \operatorname{hyp}(n, x_{l-1} - x_l, \omega_l \epsilon)$$

Let's approximate now with

$$\operatorname{hyp}(n, j, \omega_l \epsilon) \approx \begin{cases} \dfrac{\omega_l^j \epsilon^j}{j!}, & 0 \le j < n \\ \dfrac{\omega_l^{n+j} \epsilon^{n+j}}{(n+j)!}, & -n < j < 0 \end{cases}$$

Then

$$a_{x_l x_{l-1}} = \begin{cases} \frac{\epsilon^{x_{l-1} - x_l}}{(x_{l-1} - x_l)!}, & x_{l-1} \ge x_l \\ \frac{\omega^n \epsilon^{n + x_{l-1} - x_l}}{(n + x_{l-1} - x_l)!}, & x_{l-1} < x_l \end{cases}$$

Finally, we organize the trace into orders of ϵ. Only $\epsilon^0, \epsilon^n, \epsilon^{2n}, \dots$ terms exist. The number of ways to have $\omega_{a_1}^n \cdot \omega_{a_2}^n \cdot \dots \omega_{a_m}^n$ is given by

$$P = \sum_{i_1 = a_1}^{a_2} \sum_{i_2 = i_1}^{a_2} \dots \sum_{i_n = i_{n-1}}^{a_2} (1) \cdot \sum_{i_1 = a_2}^{a_3} \sum_{i_2 = i_1}^{a_3} \dots \sum_{i_n = i_{n-1}}^{a_3} (1) \cdot \dots \cdot \sum_{i_1 = a_m}^{k + a_1} \sum_{i_2 = i_1}^{k + a_1} \dots \sum_{i_n = i_{n-1}}^{k + a_1} (1)$$

$$= \frac{1}{\epsilon^{mn}} \left(\int_{a_1}^{a_2} \dots \int_{\tau_{n-1}}^{a_2} (1) \, \mathrm{d}\tau_n \dots \mathrm{d}\tau_1 \cdot \dots \right) = \frac{(a_2 - a_1)^{n-1}}{(n-1)!} \frac{(a_3 - a_2)^{n-1}}{(n-1)!} \dots \frac{(k + a_1 - a_m)^{n-1}}{(n-1)!}$$

Thus the trace of the solution matrix over a period for the ODE of the form of (14) is given by:

$$\operatorname{tr}\mathbf{G} = n + \frac{1}{(n-1)!}\int_{\tau_0}^{T+\tau_0} T^{n-1}\,\mathrm{d}\tau_1 \tag{17}$$
$$+ \frac{1}{(n-1)!^2}\int_{\tau_0}^{T+\tau_0}\int_{\tau_1}^{T+\tau_0} (T+\tau_1-\tau_2)^{n-1}(\tau_2-\tau_1)^{n-1}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$
$$+ \frac{1}{(n-1)!^3}\int_{\tau_0}^{T+\tau_0}\int_{\tau_1}^{T+\tau_0}\int_{\tau_2}^{T+\tau_0} (T+\tau_1-\tau_3)^{n-1}(\tau_3-\tau_2)^{n-1}(\tau_2-\tau_1)^{n-1}\,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \ldots$$

Unfortunatly, this does not have as much power for ODEs higher than second order. The eigenvalues of second order ODEs can be described fully by the determinant and trace, and the determinant is known, so the trace of the period map fully describes stability. However, the eigenvalues of third or higher order ODEs cannot be described fully by the determinant and trace, and so we cannot acquire a stability condition with this trace alone.

4.5 Applications of the Integral Summation

Returning to second order ODEs, the integral summation can be calculated directly in a select few cases, such as the case $f(\tau) = k$, where k is constant. The summation is calculated by induction.

First, take the nth integral term. τ_0 is $-T$ so that the upper bounds of the integrals will be zero, and T is any positive number, since k is periodic of any period.

$$I = \int_{-T}^{0}\int_{\tau_1}^{0}\cdots\int_{\tau_{n-1}}^{0} (T+\tau_1-\tau_n)(\tau_n-\tau_{n-1})\ldots(\tau_2-\tau_1)k^n\,\mathrm{d}\tau_n\ldots\mathrm{d}\tau_2\,\mathrm{d}\tau_1$$

After integrating $n-m-1$ times, the integral will be something like

$$I = \int_{-T}^{0}\cdots\int_{\tau_m}^{0} (C_{m+1}^{p}\tau_{m+1}^{p} + \ldots + C_{m+1}^{1}\tau_{m+1} + C_{m+1}^{0})(\tau_{m+1}-\tau_m)\ldots k^n\,\mathrm{d}\tau_{m+1}\ldots\mathrm{d}\tau_1$$

Where C_{m+1}^{p} is the constant multiplying τ_{m+1}^{p}. Taking the next integral,

$$I = \int_{-T}^{0}\cdots\int_{\tau_m}^{0} (C_{m+1}^{p}\tau_{m+1}^{p+1} + \ldots + C_{m+1}^{1}\tau_{m+1} + C_{m+1}^{0}$$
$$- C_{m+1}^{p}\tau_{m+1}^{p}\tau_i - \ldots - C_{m+1}^{1}\tau_{m+1}\tau_m - C_{m+1}^{0})k^n\,\mathrm{d}\tau_{m+1}\ldots\mathrm{d}\tau_1$$

$$= \int_{-T}^{0}\cdots\int_{\tau_{m-1}}^{0}\left(\left(\frac{1}{p+1}-\frac{1}{p+2}\right)C_{m+1}^{p}\tau_m^{p+2} + \ldots\right.$$
$$\left. + \left(\frac{1}{2}-\frac{1}{3}\right)C_{m+1}^{1}\tau_m^{3} + \left(1-\frac{1}{2}\right)C_{m+1}^{0}\tau_m^{2}\right)k^n\,\mathrm{d}\tau_m\ldots\mathrm{d}\tau_1$$

So

$$C_m^{p+2} = \left(\frac{1}{p+1}-\frac{1}{p+2}\right)C_{m+1}^{p}$$

And

$$C_n^0 = T + x_1, \quad C_n^1 = -1$$

The final integration is

$$I = \int_{-T}^{0} (C_1^{2n-2}\tau_1^{2n-2} + C_1^{2n-1}\tau_1^{2n-1})k^n \,\mathrm{d}\tau_1$$

$$C_1^{2n-1} = \left(\frac{1}{2n-2} - \frac{1}{2n-1}\right)\left(\frac{1}{2n-4} - \frac{1}{2n-3}\right)\dots\left(\frac{1}{2} - \frac{1}{3}\right) C_n^1$$

$$C_1^{2n-2} = \left(\frac{1}{2n-3} - \frac{1}{2n-2}\right)\left(\frac{1}{2n-5} - \frac{1}{2n-4}\right)\dots\left(1 - \frac{1}{2}\right) C_n^0$$

$$C_1^{2n-1} = -\frac{1}{(2n-1)!}, \quad C_1^{2n-2} = \frac{1}{(2n-2)!}(T + \tau_1)$$

$$I = \int_{-T}^{0} \left(\frac{1}{(2n-2)!}(T+\tau_1)\tau_1^{2n-2} - \frac{1}{(2n-1)!}\tau_1^{2n-1}\right) k^n \,\mathrm{d}\tau_1$$

$$= \int_{-T}^{0} \left(\frac{T}{(2n-2)!}\tau_1^{2n-2} + \frac{2n-2}{(2n-1)!}\tau_1^{2n-1}\right) k^n \,\mathrm{d}\tau_1$$

$$= \frac{T^{2n}}{(2n-1)!} + \frac{(2-2n)T^{2n}}{(2n)!}k^n = 2\frac{T^{2n}}{(2n)!}k^n$$

This is the value of the nth integral in the summation. If the integrals are now summed, a familiar expression arises.

$$\sum_{n=0}^{\infty} 2\frac{T^{2n}k^n}{(2n)!} = 2\cosh(T\sqrt{k})$$

This is less than two if the argument $2\pi\sqrt{k}$ is imaginary, and is greater than two if the argument is real. So the equation is stable if $k < 0$ and unstable if $k > 0$. In actuality, this equation is autonomous, so can be placed in the Trace Determinant Plane. Its place is on the Det-axis, either a harmonic oscillator or a saddle. A harmonic oscillator (stable) if $k < 0$ and a saddle (unstable) if $k > 0$.

A more difficult application of the integral summation (13) is the square wave equation. The process, while more complicated, uses the same basic idea. First take the nth integral term, setting $\tau_0 = -\pi$ and $T = 2\pi$.

$$I = \int_{-\pi}^{\pi} \dots \int_{\tau_{n-1}}^{\pi} (2\pi + \tau_1 - \tau_n)\dots(\tau_2 - \tau_1)(\beta - \alpha \,\mathrm{squ}\,\tau_n)\dots(\beta - \alpha \,\mathrm{squ}\,\tau_1) \,\mathrm{d}\tau_n \dots \mathrm{d}\tau_1$$

The squ functions can be taken out to simplify the integral, changing it to

$$I = \sum_{j=0}^{n} R_j$$

where

$$R_j = \int_{-\pi}^{0} \dots \int_{\tau_{j-2}}^{0} \int_{0}^{\pi} \int_{\tau_j}^{\pi} \dots \int_{\tau_{n-1}}^{\pi} (2\pi + \tau_n - \tau_1) \dots (\tau_2 - \tau_1)(\beta - \alpha)^{n-j}(\beta + \alpha)^j \, \mathrm{d}\tau_n \dots \mathrm{d}\tau_1$$

To change the upper bounds of the integrals to zero, shift every τ_i, $i \geq j$, up by π.

$$R_j = \int_{-\pi}^{0} \dots \int_{-\pi}^{0} \dots \int_{\tau_{n-1}}^{0} (\pi + \tau_1 - \tau_n) \dots (\pi + \tau_j - \tau_{j-1}) \dots (\tau_2 - \tau_1)(\beta - \alpha)^{n-j}(\beta + \alpha)^j \mathrm{d}\tau_n \dots \mathrm{d}\tau_1$$

The integration should be separated into four parts: $i > j$, $i = j$, $j > i > 1$, $i = 1$. For the first and third parts, the integration is as simple as that of the earlier application. All that is left is to begin.

$$C_m^p = \frac{1}{p(p-1)} C_{m+1}^{p-2}$$

Let $l = n - j$

$$I = \int_{-\pi}^{0} \dots \int_{\tau_{j-2}}^{0} \int_{-\pi}^{0} (C_j^{2l+1} \tau_j^{2l+1} + C_j^{2l} \tau_j^{2l})(\pi + \tau_j - \tau_{j-1}) \dots (\beta - \alpha)^l (\beta + \alpha)^j \, \mathrm{d}\tau_j \dots \mathrm{d}\tau_1$$

$$= \int_{-\pi}^{0} \dots \int_{\tau_{j-2}}^{0} \int_{-\pi}^{0} \left(\frac{C_n^1}{(2l+1)!} \tau_j^{2l+1} + \frac{C_n^0}{(2l)!} \tau_j^{2l} \right) (\pi + \tau_j - \tau_{j-1}) \dots (\beta - \alpha)^l (\beta + \alpha)^j \, \mathrm{d}\tau_j \dots \mathrm{d}\tau_1$$

$$= \int_{-\pi}^{0} \dots \int_{\tau_{j-2}}^{0} \left(\frac{C_n^0 \pi^{2l+2}}{(2l+2)!} - \frac{C_n^1 \pi^{2l+3}}{(2l+3)!} - \left(\frac{C_n^0 \pi^{2l+1}}{(2l+1)!} - \frac{C_n^1 \pi^{2l+2}}{(2l+2)!} \right) \tau_{j-1} \right) \dots (\beta - \alpha)^l (\beta + \alpha)^j \mathrm{d}\tau_{j-1} \dots \mathrm{d}\tau_1$$

$$C_{j-1}^0 = \frac{(2l+4)\pi^{2l+3}}{(2l+3)!} + \frac{\pi^{2l+2}\tau_1}{(2l+2)!} \qquad C_{j-1}^1 = -\frac{(2l+3)\pi^{2l+2}}{(2l+2)!}\tau_{j-1} - \frac{\pi^{2l+1}\tau_1}{(2l+1)!}\tau_{j-1}$$

$$R_j = \int_{-\pi}^{0} (C_1^{2j-4} x_1^{2j-4} + C_1^{2l-3} x_1^{2l-3})(\beta - \alpha)^l (\beta + \alpha)^j \, \mathrm{d}x_1$$

$$R_j = \pi^{2n} \left(\frac{2l+4}{(2j-3)!(2l+3)!} - \frac{2j-3}{(2j-2)!(2l+2)!} + \frac{2l+3}{(2j-2)!(2l+2)!} - \frac{2l-2}{(2j-1)!(2l+1)!} \right)$$

After simplification, this becomes

$$R_j = \frac{\pi^{2n}(2n+1)(2n+2)}{(2l+1)!(2j+1)!}(\beta-\alpha)^l(\beta+\alpha)^j$$

So the nth integral term is

$$I = \sum_{j=0}^{n} \frac{\pi^{2n}(2n+1)(2n+2)}{(2n-2j+1)!(2j+1)!}(\beta-\alpha)^{n-j}(\beta+\alpha)^j$$

Summing over n gives the trace:

$$\operatorname{tr}(\mathbf{A}_{-\pi}^{\pi}) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{\pi^{2n}(2n+1)(2n+2)}{(2n-2j+1)!(2j+1)!}(\beta-\alpha)^{n-j}(\beta+\alpha)^j$$

Finally, switching the sums and simplifying gives:

$$\begin{aligned}
\operatorname{tr}(\mathbf{A}_{-\pi}^{\pi}) = &\sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \frac{2\pi^{2j}\pi^{2n}}{(2j)!(2n)!} \\
&+ \sum_{j=0}^{\infty} \frac{\pi^{2j-1}}{(2j-1)!}\sqrt{\beta-\alpha}^{2j-1} \sum_{n=0}^{\infty} \frac{\pi^{2n+1}}{(2n+1)!}\sqrt{\beta+\alpha}^{2n+1} \frac{\sqrt{\beta-\alpha}}{\sqrt{\beta+\alpha}} \\
&+ \sum_{j=0}^{\infty} \frac{\pi^{2j+1}}{(2j+1)!}\sqrt{\beta-\alpha}^{2j+1} \sum_{n=0}^{\infty} \frac{\pi^{2n-1}}{(2n-1)!}\sqrt{\beta+\alpha}^{2n-1} \frac{\sqrt{\beta+\alpha}}{\sqrt{\beta-\alpha}} \\
= &\; c_1c_2 + s_1s_2\frac{\omega_1}{\omega_2} + s_1s_2\frac{\omega_2}{\omega_1} \\
= &\; c_1c_2 + s_1s_2\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)
\end{aligned} \tag{18}$$

This is the same value attained using the earlier method.

Obviously this method works. Its power, however, is not in finding a solution for these almost trivial cases, but in approximating, perhaps even finding an exact solution, for more difficult cases such as Mathieu's Equation.

For Mathieu's Equation, the trace is not exactly solved. It can be approximated with any amount of precision, simply by evaluating further integrals. The first two are presently solved.

($\operatorname{tr}_{n\omega}$ is the integral term in the trace including n ωs.)

$$\operatorname{tr}_{1\omega} = \int_{-\pi}^{\pi} 2\pi(\beta - \alpha\cos a)\,\mathrm{d}a = [2\pi\beta a - \alpha \sin a]_{-\pi}^{\pi} = 4\pi^2\beta$$

$$
\begin{aligned}
\operatorname{tr}_{2\omega} &= \int_{-\pi}^{\pi}\int_{a}^{\pi} \big(2\pi-(b-a)\big)(b-a)(\beta-\alpha\cos b)(\beta-\alpha\cos b)\,\mathrm{d}b\,\mathrm{d}a \\
&= \int_{-\pi}^{\pi}\big[\pi\beta b^2 - 2\pi\beta ab - \tfrac{1}{3}\beta b^3 - \beta a^2 b + \beta ab^2 - 2\pi\alpha b\sin b - 2\pi\alpha\cos b \\
&\qquad + 2\pi a\sin b + \alpha b^2\sin b + 2\alpha b\cos b - 2\alpha\sin b - \alpha a^2\sin b \\
&\qquad + 2\alpha ab\sin b + 2\alpha a\cos b\big]_a^{\pi}(\beta-\alpha\cos a)\,\mathrm{d}a \\
&= \int_{-\pi}^{\pi}\big(\tfrac{2}{3}\pi^3\beta + 2\pi\alpha\cos a + \text{odd functions}\big)(\beta-\alpha\cos a)\,\mathrm{d}a = \tfrac{4}{3}\pi^4\beta^2 - 2\pi^2\alpha^2
\end{aligned}
$$

$$
= \tfrac{4}{3}\pi^4\beta^2 - 2\pi^2\alpha^2
$$

As can be seen, calculating the integrals very quickly becomes difficult as more ωs are taken into account. Fortunately, computing only the first two of the integrals gives a reasonably accurate description of stability for a limited range of α and β. With further computation, the next two integrals may be evaluated:

$$
\operatorname{tr}_{3\omega} = \tfrac{8}{45}\pi^6\beta^3 - (\tfrac{4}{3}\pi^4 - 8\pi^2)\alpha^2\beta
$$

$$
\operatorname{tr}_{4\omega} = (\tfrac{1}{3}\pi^4 - \tfrac{25}{8}\pi^2)\alpha^4 - (\tfrac{4}{15}\pi^6 - \tfrac{16}{3}\pi^4 + 32\pi^2)\alpha^2\beta^2 + \tfrac{4}{315}\pi^8\beta^4
$$

These were computed using Mathematica, as were the integrals up to $\operatorname{tr}_{10\omega}$, though these were far to large to include here.

So for limited α and β, the linear ODE (5) is stable if the following condition is met:

$$
2 > \left|2 + 4\pi^2\beta + \tfrac{4}{3}\pi^4\beta^2 - 2\pi^2\alpha^2\right|
$$

Or, for greater accuracy and larger domain:

$$
\begin{aligned}
2 > \Big|2 &+ 4\pi^2\beta + \tfrac{4}{3}\pi^4\beta^2 - 2\pi^2\alpha^2 + \tfrac{8}{45}\pi^6\beta^3 - (\tfrac{4}{3}\pi^4 - 8\pi^2)\alpha^2\beta \\
&+ (\tfrac{1}{3}\pi^4 - \tfrac{25}{8}\pi^2)\alpha^4 - (\tfrac{4}{15}\pi^6 - \tfrac{16}{3}\pi^4 + 32\pi^2)\alpha^2\beta^2 + \tfrac{4}{315}\pi^8\beta^4\Big|
\end{aligned} \tag{19}
$$

This is a fair way to approximate the stability, but there is more than one way to do so. Using iterative integration once again, the condition for stability can be approximated as a Taylor Series for just α, as opposed to both α and β. Of course, the Taylor's Series is dependant only on even powers of α, since a sign change of α can be thought of as a phase change of τ, and so should not affect the stability.

Mathieu's Equation has $f(\tau) = \beta - \alpha\cos\tau$. In the previous two applications, a term being integrated in some integral of the nth term of the trace summation could be fully described by the subscript of the variable being integrated over, m, and the power of the variable, p. In application to Mathieu's Equation, however, the $\cos\tau$ is a severe complication.

First, set $T = 2\pi$ and $\tau_0 = -2\pi$ in order for the upper limits of the integrals to be 0, and define $k0 = 2\pi + \tau_1$, $k1 = -1$ as the initial coefficients. Also, it simplifies the problem to change $\cos\tau$ to its exponential form and to define $\epsilon = -\frac{1}{2}\alpha$. The nth term in the trace summation then becomes:

$$I = \int_{-2\pi}^{0} \dots \int_{\tau_{n-1}}^{0} (2\pi + \tau_1 - \tau_n) \dots (\tau_2 - \tau_1)(\beta + \epsilon e^{-i\tau_n} + \epsilon e^{i\tau_n}) \dots (\beta + \epsilon e^{-i\tau_1} + \epsilon e^{i\tau_1})\, \mathrm{d}\tau_n \dots \mathrm{d}\tau_1$$

And a term can be fully described by four numbers: the variable power, p, the power of the exponential, r, the power of ϵ, z, and the variable subscript, m. Define $l = n - m$ and $s = l - r = n - m - r$ to be used later. Let's integrate an arbitrary term:

$$F = \int_{\tau_{m-1}}^{0} {}^{r}_{z}C^{p}_{m} \tau_m^p e^{ir\tau_m} (\tau_m - \tau_{m-1})(\beta + \epsilon e^{i\tau_n} + \epsilon e^{-i\tau_n})\, \mathrm{d}\tau_m$$

$$= {}^{r}_{z}C^{p}_{m} \int_{\tau_{m-1}}^{0} (\tau_m^{p+1} - \tau_m^p \tau_{m-1})(\beta e^{ir\tau_m} + \epsilon e^{i(r-1)\tau_m} + \epsilon e^{i(r+1)\tau_m})$$

Since for $r \neq 0$,

$$\int_{y}^{0} x^p e^{irx}\, \mathrm{d}x = \sum_{q=0}^{p} \left(\frac{i}{r}\right)^{p-q+1} \frac{p!}{q!} y^q e^{irx} - \left(\frac{i}{r}\right)^{p+1} p!,$$

$$\begin{aligned} F = {}^{r}_{z}C^{p}_{m} \Bigg[&\sum_{j=0}^{p+1} \left(1 - \frac{j}{p+1}\right) \frac{(p+1)!}{j!} \tau_{m-1}^{j} \\ &\cdot \left(\beta e^{ir\tau_{m-1}} \left(\frac{i}{r}\right)^{p+2-j} + \epsilon e^{i(r-1)\tau_{m-1}} \left(\frac{i}{r-1}\right)^{p+2-j} + \epsilon e^{i(r+1)\tau_{m-1}} \left(\frac{i}{r+1}\right)^{p+2-j}\right) \\ &+ (\tau_{m-1} - (p+1)) \left(\beta \left(\frac{i}{r}\right)^{p+1} + \epsilon \left(\frac{i}{r+1}\right)^{p+1} + \epsilon \left(\frac{i}{r-1}\right)^{p+1}\right) p! \Bigg] \end{aligned}$$

when $r \neq 0, -1, 1$. When $r = 0, -1,$ or 1, remove the undefined terms and add onto the end

$${}^{r}_{z}C^{p}_{m} \frac{\beta \text{ (or } \epsilon)}{(p+1)(p+2)} \tau_{m-1}^{p+2}$$

Taking these, we can find a term in the mth integral of the nth set of integrals of the trace summation from terms in the $(m+1)$th integral. There are four distinct forms for this relation:

$${}^{r}_{z}C^{p}_{m} = \sum_{q=0}^{2s-p+1} \left(\frac{(p+1)! - q(p)!}{q!}\right) \left(\frac{i}{r}\right)^{p-q+2} \left(\beta\, {}^{r}_{z}C^{q}_{m+1} + \epsilon\, {}^{r-1}_{z-1}C^{q}_{m+1} + \epsilon\, {}^{r+1}_{z-1}C^{q}_{m+1}\right), \quad r \neq 0 \tag{20}$$

$$ {}^{0}_{z}C^{p}_{m} = \frac{1}{p(p-1)}\left(\beta\, {}^{0}_{z}C^{p-2}_{m+1} + \epsilon\, {}^{-1}_{z-1}C^{p-2}_{m+1} + \epsilon\, {}^{1}_{z-1}C^{p-2}_{m+1}\right), \quad p \geq 2 \tag{21} $$

$$ {}^{0}_{z}C^{1}_{m} = 2\mathrm{Re}\left[\sum_{r=1}^{z}\sum_{p=0}^{2s+1} p!\left(\frac{i}{r}\right)^{p+1}\left(\beta\, {}^{r}_{z}C^{p}_{m+1} + \epsilon\, {}^{r-1}_{z-1}C^{p}_{m+1} + \epsilon\, {}^{r+1}_{z-1}C^{p}_{m+1}\right)\right] \tag{22} $$

$$ {}^{0}_{z}C^{0}_{m} = -2\mathrm{Re}\left[\sum_{r=1}^{z}\sum_{p=0}^{2s+1} (p+1)!\left(\frac{i}{r}\right)^{p+2}\left(\beta\, {}^{r}_{z}C^{p}_{m+1} + \epsilon\, {}^{r-1}_{z-1}C^{p}_{m+1} + \epsilon\, {}^{r+1}_{z-1}C^{p}_{m+1}\right)\right] \tag{23} $$

Where $s = l - r = n - m - r$. Note that for $p > 2s+1$, ${}^{r}_{z}C^{p}_{m} = 0$ (integration raises the power only so fast). Remember that the exponentials all came from trigonometric functions, and so they must add together at each step to give trigonometric functions once again. Thus ${}^{-r}_{z}C^{p}_{m}$ is the complex conjugate of ${}^{r}_{z}C^{p}_{m}$, so only one need be found. From now on, assume $r \geq 0$ unless otherwise stated.

The object is to acquire the trace of the period map in the form of a Taylor Series for ϵ. We divide the work by the value of $z - |r|$. If $z' - |r'| > z - |r|$, then ${}^{r'}_{z'}C^{m'}_{p'}$ cannot have any effect upon ${}^{r}_{z}C^{p}_{m}$. So we will begin at $z - r = 0$, $r \geq 0$. First of all,

$$ {}^{0}_{0}C^{p}_{m} = \frac{\beta}{p(p-1)}\, {}^{0}_{0}C^{p-2}_{m+1} = \begin{cases} \frac{\beta^{\frac{p}{2}}}{p!}k_0, & p \text{ even} \\ \frac{\beta^{\frac{p-1}{2}}}{p!}k_1, & p \text{ odd} \end{cases} $$

since ${}^{-1}_{-1}C^{p-2}_{m+1} = {}^{1}_{-1}C^{p-2}_{m+1} = 0$. So the α^0 part of the trace is the simple harmonic oscillator encountered previously. Thus we have

$$ \mathrm{tr}\,(\mathbf{A}^{0}_{-2\pi}) = 2\cosh(2\pi\sqrt{\beta}) + O(\alpha^2) $$

Now we proceed to $z = r$, $r > 0$.

Proposition

$$ {}^{r}_{r}C^{p}_{m} = (-1)^r \frac{\epsilon^r \beta^{l-r}}{r!^2 p!}\left[(2i)^{2s-p}c_r^{(2s-p)} + (2i)^{2s-p+1}c_r^{(2s-p+1)}\right], \tag{24} $$

$$ c_r^{(n)} = \sum_{x_1=1}^{r}\frac{1}{x_1}\sum_{x_2=1}^{x_1}\frac{1}{x_2}\cdots\sum_{x_n=1}^{x_{n-1}}\frac{1}{x_n}. \tag{25} $$

The number $c_r^{(n)}$ is known as the rth harmonic number of order n. For more on harmonic numbers, see [4]. The natural way to write it is as given, but at times we will employ the following identity to simplify:

$$ c_r^{(n)} = \sum_{i=1}^{r}\binom{r}{i}\frac{(-1)^{i-1}}{i^n} $$

pf:
Proof by induction. Suppose that for all p', r', and $m' > m$, we have

$${}^{r'}_{r'}C^{p'}_{m'} = (-1)^{r'} \frac{\epsilon^{r'} \beta^{l'-r'}}{r'!^2 p'!} \left[(2i)^{2s'-p'} c^{(2s'-p')}_{r'} + (2i)^{2s'-p'+1} c^{(2s'-p'+1)}_{r'} \right]$$

From (20), we have

$${}^{r}_{r}C^{p}_{m} = \sum_{q=0}^{2s+1-p} (q+1) \frac{(p+q)!}{p!} \left(\frac{i}{r}\right)^{q+2} \left(\beta \, {}^{r}_{z}C^{p+q}_{m+1} + \epsilon \, {}^{r-1}_{r-1}C^{p+q}_{m+1} + \epsilon \, {}^{r+1}_{r-1}C^{p+q}_{m+1} \right)$$

The power of the exponential can never be greater than the power of ϵ, so ${}^{r+1}_{r-1}C^{p+q}_{m+1} = 0$. For $q = 2s-p, 2s-p+1$, ${}^{r}_{r}C^{p+q}_{m+1} = 0$ because then $p' > 2s'+1$. Define $j = 2s-p+1$.

$${}^{r}_{r}C^{2s-j+1}_{m} = \sum_{q=0}^{j} (q+1) \frac{(2s-j+1+q)!}{(2s-j+1)!} \left(\frac{i}{r}\right)^{q+2} \left(\beta \, {}^{r}_{r}C^{2s-j+1+q}_{m+1} + \epsilon \, {}^{r-1}_{r-1}C^{2s-j+1+q}_{m+1} \right)$$

$$= \sum_{q=0}^{j} (q+1) \frac{(2s-j+1+q)!}{(2s-j+1)!} \left(\frac{i}{r}\right)^{q+2} \left(\epsilon \, {}^{r-1}_{r-1}C^{2s-j+1+q}_{m+1} \right) + \sum_{q=0}^{j-2} (q+1) \frac{(2s-j+1+q)!}{(2s-j+1)!} \left(\frac{i}{r}\right)^{q+2} \left(\beta \, {}^{r}_{r}C^{2s-j+1+q}_{m+1} \right)$$

$$= (-1)^r \frac{\epsilon^r \beta^{l-r}}{r!^2 p!} \Bigg(-\sum_{q=0}^{j} (q+1) \left(\frac{i}{r}\right)^{q+2} r^2 \left[(2i)^{j-q-1} c^{(j-q-1)}_{r-1} + (2i)^{j-q} c^{(j-q)}_{r-1} \right] + \sum_{q=0}^{j-2} (q+1) \left(\frac{i}{r}\right)^{q+2} \left[(2i)^{j-q-3} c^{(j-q-3)}_{r} + (2i)^{j-q-2} c^{(j-q-2)}_{r} \right] \Bigg)$$

$$= (-1)^r \frac{\epsilon^r \beta^{l-r}}{r!^2 p!} \Bigg((2i)^{j-1} k_0 \left[\sum_{q=0}^{j-2} (q+1) \left(\frac{i}{r}\right)^{q+2} (2i)^{-q-2} c^{(j-q-3)}_{r} + \sum_{q=0}^{j} (q+1) \left(\frac{i}{r}\right)^{q} (2i)^{-q} c^{(j-q-1)}_{r-1} \right] + (2i)^{j} k_1 \left[\sum_{q=0}^{j-2} (q+1) \left(\frac{i}{r}\right)^{q+2} (2i)^{-q-2} c^{(j-q-2)}_{r} + \sum_{p=0}^{j} (q+1) \left(\frac{i}{r}\right)^{q} (2i)^{-q} c^{(j-q)}_{r-1} \right] \Bigg)$$

Thus, it will be sufficient to prove that

$$c^{(j)}_{r} = \sum_{q=0}^{j-2} (q+1) \left(\frac{i}{r}\right)^{q+2} (2i)^{-q-2} c^{(j-q-2)}_{r} + \sum_{q=0}^{j} (q+1) \left(\frac{i}{r}\right)^{q} (2i)^{-q} c^{(j-q)}_{r-1}$$

Using the fact that

$$c_{r-1}^{(n)} = \sum_{x_1=1}^{r-1} \frac{1}{x_1} \cdots \sum_{x_n=1}^{x_{n-1}} \frac{1}{x_n} = c_r^{(n)} - \frac{1}{r} c_r^{(n-1)},$$

We have

$$\left[\sum_{q=2}^{j} (q-1) \left(\frac{i}{r}\right)^q (2i)^{-q} c_r^{(j-q)} + \sum_{q=0}^{j} (q+1) \left(\frac{i}{r}\right)^q (2i)^{-q} \left(c_r^{(j-q)} - \frac{1}{r} c_r^{(j-q-1)} \right) \right]$$

$$= \left[\sum_{q=2}^{j} (q-1) \left(\frac{1}{2r}\right)^q c_r^{(j-q)} + \sum_{q=0}^{j} (q+1) \left(\frac{1}{2r}\right)^q c_r^{(j-q)} - 2 \sum_{q=1}^{j} (q) \left(\frac{1}{2r}\right)^q c_r^{(j-q)} \right]$$

$$= c_r^{(j)}$$

By induction, because

$${}_0^0C_n^0 = k_0 = (-1)^0 \frac{\epsilon^0 \beta^0}{0!^2 0!} \left[(2i)^0 c_0^{(0)} k_0 + (2i)^1 c_0^{(1)} k_1 \right]$$

$${}_0^0C_n^1 = k_1 = (-1)^0 \frac{\epsilon^0 \beta^0}{0!^2 1!} \left[(2i)^{-1} c_0^{(-1)} k_0 + (2i)^0 c_0^{(0)} k_1 \right],$$

it is true that

$${}_r^rC_m^p = (-1)^r \frac{\epsilon^r \beta^{l-r}}{r!^2 p!} \left[(2i)^{2s-p} c_r^{(2s-p)} + (2i)^{2s-p+1} c_r^{(2s-p+1)} \right] \square$$

The $r = 1$ and $r = 2$ values for this will be most useful to us later. Using the fact that $c_1^{(n)} = 1$, $c_2^{(n)} = \binom{2}{1} \frac{1}{1^n} - \binom{2}{2} \frac{1}{2^n} = \frac{2^{n+1}-1}{2^n}$ for $n \geq 0$ and $c_r^{(n)} = 0$ for $n < 0$, we have

$${}_1^1C_m^{2l-j-1} = -\frac{\epsilon \beta^{l-1}}{(2l-j-1)!} \left(\delta_{j0} (2i)^{j-1} k0 + (2i)^j k1 \right)$$

$${}_2^2C_m^{2l-j-3} = \frac{\epsilon^2 \beta^{l-2}}{(2l-j-3)!} \left((2i)^{j-1} \frac{2^j - 1}{2^{j-1}} k0 + (2i)^j \frac{2^{j+1} - 1}{2^j} k1 \right)$$

With these in hand, we now find ${}_1^0C_m^p$. First, ${}_1^0C_m^1$ and ${}_1^0C_m^0$. Using (22),

$${}_1^0C_m^1 = 2\mathrm{Re} \left[\sum_{p=0}^{2l-1} p! i^{p+1} \left(\beta \, {}_1^1C_{m+1}^p + \epsilon \, {}_0^0C_{m+1}^p \right) \right]$$

$$= 2\mathrm{Re} \left[\sum_{p=0}^{2l-3} p! i^{p+1} \left(-\frac{\epsilon \beta^{l-1}}{p!} \left((2i)^{2l-p-4} k0 + (2i)^{2l-p-3} k1 \right) \right) \right.$$

$$\left. + \sum_{p=0}^{2l-1} p! i^{p+1} \left(\frac{\epsilon \beta^{l-1}}{(2l-2)!} k0 \delta_{p(2l-2)} + \frac{\epsilon \beta^{l-1}}{(2l-1)!} k1 \delta_{p(2l-1)} \right) \right]$$

$$= 2\epsilon\beta^{l-1}(-1)^l k1\left[\sum_{p=0}^{2l-3}(2^{2l-p-3})+1\right]$$

$$= (-1)^l\epsilon\beta^{l-1}\cdot 2^{2l-1}k1$$

Similarly using (23),

$${}^{0}_{1}C^{0}_{m} = (-1)^{l-1}\epsilon\beta^{l-1}\cdot 2^{2l-1}k0$$

Now using (21), for j odd,

$${}^{0}_{1}C^{j}_{m} = \frac{1}{j(j-1)}\left(\beta\, {}^{0}_{1}C^{j-2}_{m+1}\right) = \ldots = \frac{\beta^{\frac{j-1}{2}}}{j!}\, {}^{0}_{1}C^{1}_{m+\frac{j-1}{2}}$$

$$= (-1)^{l-\frac{j-1}{2}}\frac{\epsilon\beta^{l-1}}{j!}\cdot 2^{2l-j}k1$$

The coefficient for j even is found in the same manner. So

$${}^{0}_{1}C^{j}_{m} = \frac{\epsilon\beta^{l-1}}{j!}\begin{cases}(-1)^{l-\frac{j-1}{2}}\cdot 2^{2l-j}, & j \text{ odd}\\ (-1)^{l-\frac{j}{2}-1}\cdot 2^{2l-j-1}, & j\text{even}\end{cases} \tag{26}$$

At this point, we have all of the coefficients for terms with ϵ^1 and lower power, so we could integrate the appropriate terms and find the coefficient of ϵ in the Taylor Series expansion of the trace. However, recognizing that the Taylor Series only includes even powers of ϵ, this is unnecessary. The next two coefficients are given, proof left to the reader.

$${}^{r+1}_{r}C^{j}_{m} = (-1)^{r+1}\frac{\epsilon^2\beta^{l-2}\cdot 2}{r!^2 j!} \tag{27}$$
$$\begin{cases}\left[-(2i)^{2s-j-2}\sum_{t=0}^{s-\frac{j+1}{2}-1}c_r^{(2t+1)}k0 + (2i)^{2s-j-1}\sum_{t=0}^{s-\frac{j+1}{2}}c_r^{(2t)}k1\right], & j \text{ odd}\\ \left[-(2i)^{2s-j-2}\sum_{t=0}^{s-\frac{j}{2}-1}c_r^{(2t)}k0 + (2i)^{2s-j-1}\sum_{t=0}^{s-\frac{j}{2}-1}c_r^{(2t)}k1\right], & j \text{ even}\end{cases}$$

$${}^{2}_{0}C^{j}_{m} = \frac{\epsilon^2\beta^{l-2}}{j!}\begin{cases}(-1)^{l-\frac{j-1}{2}}\left[(l+\frac{j}{2}-\frac{3}{4})4^{l-\frac{j-1}{2}-1}-\frac{1}{2}-\frac{1}{4}\delta_{j(2l-1)}\right]k1, & j \text{ odd}\\ (-1)^{l-\frac{j}{2}}\left[(l+\frac{j}{2}-\frac{5}{4})4^{l-\frac{j}{2}-1}+\frac{1}{2}-\frac{1}{4}\delta_{j(2l-2)}\right]k0, & j \text{ even}\end{cases} \tag{28}$$

Finally, we sum and integrate the appropriate terms to get the part of the trace that is multiplied by ϵ^2.

$$\mathrm{tr}_{\epsilon^2} = \sum_{n=0}^{\infty} \int_{-2\pi}^{0} \Bigg(\beta \sum_{j=0}^{2n-3} {}^{0}_{2}C_1^j \tau_1^j + \beta \sum_{j=0}^{2n-5} 2\mathrm{Re}({}^{1}_{2}C_1^j \tau_1^j e^{i\tau_1}) + \beta \sum_{j=0}^{2n-5} 2\mathrm{Re}({}^{2}_{2}C_1^j \tau_1^j e^{2i\tau_1})$$

$$+2\epsilon \cos\tau_1 \sum_{j=0}^{2n-3} {}^{0}_{1}C_1^j \tau_1^j + 2\epsilon\cos\tau_1 \sum_{j=0}^{2n-3} 2\mathrm{Re}({}^{1}_{1}C_1^j \tau_1^j e^{i\tau_1}) \Bigg) \mathrm{d}\tau_1$$

After much manipulation, this has a remarkably simple form:

$$\mathrm{tr}_{\epsilon^2} = \sum_{n=0}^{\infty} \Bigg(-2\epsilon^2 \beta^n \sum_{k=0}^{n} (-4)^{n-k} \frac{(2\pi)^{2k+2}}{(2k+1)!} \Bigg)$$

Switching the sums gives

$$\mathrm{tr}_{\epsilon^2} = -2\epsilon^2 \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \beta^n (-4)^{n-k} \frac{(2\pi)^{2k+2}}{(2k+1)!} = -4\pi\epsilon^2 \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \beta^n (-4)^n \cdot \beta^k \frac{(2\pi)^{2k+1}}{(2k+1)!}$$

$$= -4\pi\epsilon^2 \sum_{k=0}^{\infty} \frac{(2\pi\sqrt{\beta})^{2k+1}}{\sqrt{\beta}(2k+1)!} \sum_{n=0}^{\infty} (-4\beta)^n = -\frac{4\pi \sinh(2\pi\sqrt{\beta})}{\sqrt{\beta}(1+4\beta)}\, \epsilon^2$$

So the trace of the period mapping is

$$\mathrm{tr}\,(\mathbf{A}^0_{-2\pi}) = 2\cosh(2\pi\sqrt{\beta}) - \frac{4\pi \sinh(2\pi\sqrt{\beta})}{\sqrt{\beta}(1+4\beta)}\, \epsilon^2 + O(\epsilon^4) \tag{29}$$

Though not proven, the ϵ^4 part of the trace seems to be

$$\mathrm{tr}_{\epsilon^4} = \Bigg(\frac{4\pi^2 \cosh(2\pi\sqrt{\beta})}{\beta(1+4\beta)^2} - \frac{2\pi \sinh(2\pi\sqrt{\beta})}{\beta^{\frac{3}{2}}(1+4\beta)^2} - \frac{16\pi \sinh(2\pi\sqrt{\beta})}{\sqrt{\beta}(1+4\beta)^3} - \frac{9\pi \sinh(2\pi\sqrt{\beta})}{\sqrt{\beta}(1+\beta)(1+4\beta)^2} \Bigg) \epsilon^4$$

If α is preffered, a simple substitution here of $\epsilon = -\frac{1}{2}\alpha$ can be made. A couple of interesting things to note are

$$\mathrm{tr}_{\epsilon^2} = -\frac{2}{1+4\beta} \cdot \frac{\mathrm{d}}{\mathrm{d}\beta} \mathrm{tr}_{\epsilon^0}$$

$$\mathrm{tr}_{\epsilon^4} = -\frac{1}{1+4\beta} \cdot \frac{\mathrm{d}}{\mathrm{d}\beta} \mathrm{tr}_{\epsilon^2} - \frac{9\pi \sinh(2\pi\sqrt{\beta})}{\sqrt{\beta}(1+\beta)(1+4\beta)^2}$$

The terms exhibit enough pattern for there to perhaps be an exact equation. Anyways, these have an advantage over the noniterative approximations used above in that these are accurate for all β. There is only error in ϵ, or α.

5 Friction

Analysis of equations (2), (5), and (6) is very interesting, but to study more real systems, friction must be taken into account. Three equations will be studied.

$$\theta'' = 2\gamma\theta' + (\beta - \alpha\cos\tau)\sin\theta \tag{30}$$

$$\theta'' = 2\gamma\theta' + (\beta - \alpha\cos\tau)\theta \tag{31}$$

$$\theta'' = 2\gamma\theta' + (\beta - \alpha\,\mathrm{squ}\,\tau)\theta \tag{32}$$

The first equation is nonlinear, so once again, it will be analyzed by numerical means. The second equation is of more interest, but the third equation is easier, so it will be analyzed first. Before even this, however, it must be known how Floquet Theory applies to systems with damping.

5.1 Floquet Theory with Damping

In Section 4.3, a condition for stability for Hill's Equation was found. The goal of this section should be to find a similar condition for the damped Hill's Equation.

$$\theta'' = 2\gamma\theta' + f(\tau)\theta, \qquad f(\tau + T) = f(\tau) \tag{33}$$

It should be noted that the $2\gamma\theta'$ term only represents friction if γ is negative. If γ is positive, then energy is being added to the system, and not depleted. However, $2\gamma\theta'$is a more general way of writing the equation, and there is no reasonthat $\gamma < 0$ cannot be asserted at any time.

In Section 3.3, the condition for stability is that the trace of the solution matrix over one period is less than two. However, this is only for second order, linear, periodic, *area preserving* ODEs. This condition will not apply to an equation with a θ' term, because the area is not preserved. The first step in finding stability for the damped Hill'sEquation will be to find the new condition for Floquet Theory whenapplied to systems with friction.

Considering (33), the first three items of Theorem 1 still apply; it is only the last that requires area/volume preservation. So the mapping over one period is still $\mathbf{G}$. The difference is that $\det(\mathbf{G})$ need not necessarily equal 1.

For any equation of the form of (33), the solution matrix over a small interval of time, ϵ, is

$$\mathbf{A}_\tau^{\tau+\epsilon} = e^{\gamma\epsilon}\begin{pmatrix} c_\tau - \frac{\gamma}{\omega_\tau}s_\tau & \frac{1}{\omega_\tau}s_\tau \\ \frac{\omega_\tau^2 - \gamma^2}{\omega_\tau}s_\tau & c_\tau + \frac{\gamma}{\omega_\tau}s_\tau \end{pmatrix}$$

where

$$c_\tau = \cosh(\omega_\tau\epsilon),\ s_\tau = \sinh(\omega_\tau\epsilon),\ \omega_\tau = \sqrt{\gamma^2 + f(\tau)}$$

The solution matrix over one period is the product of $\frac{T}{\epsilon}$ of these matrices, from $\tau = x$ to $\tau = x + T$. The determinant of this product is needed. But the determinant of a product is the product of the determinants, $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$, and $\det \mathbf{A}_{\tau}^{\tau+\epsilon} = e^{\gamma\epsilon}$, so

$$\det \mathbf{A}_{x}^{x+T} = \det \mathbf{A}_{x}^{x+\epsilon} \cdot \ldots \cdot \det \mathbf{A}_{x+T-\epsilon}^{x+T} = e^{\gamma\epsilon} \cdot e^{\gamma\epsilon} \cdot \ldots = e^{\gamma T}$$

As before, the stability of the solution matrix can be determined by its trace. The real parts of the eigenvalues are what determines stability/instability. If $\mathrm{Re}\lambda_1 > 1$ or $\mathrm{Re}\lambda_2 > 1$, then the matrix is unstable. If $\mathrm{Re}\lambda_1 < 1$ and $\mathrm{Re}\lambda_2 < 1$, then it is stable. $\det \mathbf{A}_{x}^{x+T} = \lambda_1\lambda_2 = e^{\gamma T}$, so with complete generality, $\lambda_1 = e^{z}$ and $\lambda_2 = e^{\gamma T - z}$, where z is not necessarily real. Either λ_1 and λ_2 are complex conjugates, or are both real numbers, since the trace must be real. If they are complex conjugates, then $e^{\gamma T - x - iy} = e^{x-iy}$, so $x = \frac{\gamma T}{2}$, and so $\lambda_1 = e^{\gamma T + iy}$, $\lambda_2 = e^{\gamma T - iy}$. Then the trace is $\lambda_1 + \lambda_2 = 2e^{\gamma T}\cos y$. If they are real, then the trace is $\lambda_1 + \lambda_2 = e^{\gamma T - x} + e^{x}$.

Finding stability using the trace of the solution matrix must be separated into two cases: $\gamma > 0$ and $\gamma < 0$. If γ is positive, then real eigenvalues will never both be smaller than one. Since $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2$, then for complex conjugate eigenvalues, if the trace is less than two, then $\mathrm{Re}\lambda_1$ and $\mathrm{Re}\lambda_2$ are smaller than one. If γ is negative, then the real part of complex eigenvalues will always be smaller than one. For the real eigenvalues, the point where one or the other becomes greater than one is when the trace is $e^{\gamma T} + 1$. So if the trace is less than this, then the eigenvalues are both less than one.

Thus, the conditions for stability are

$$\gamma \geq 0 \quad \text{and} \quad \mathrm{tr}\,(\mathbf{A}_{x}^{x+T}) < 2, \tag{34}$$

or

$$\gamma < 0 \quad \text{and} \quad \mathrm{tr}\,(\mathbf{A}_{x}^{x+T}) < 1 + e^{\gamma T}$$

The next task is to find, like in Section 4.3, an equation that gives the trace of any matrix of the form (31).

As it turns out, the trace depends on γ only through the $e^{\gamma\epsilon}$ and through ω_τ. Inputing $\omega_\tau = \sqrt{\gamma^2 + f(\tau)}$ into the trace for the frictionless Hill's equation and multiplying by $e^{\gamma T}$ gives the trace for damped Hill's equation. The trace of the solution matrix over one period for the damped Hill's equation as written, (33), is

$$\begin{aligned} \mathrm{tr}\,(\mathbf{A}_{x}^{x+T}) = e^{\gamma T}\Bigg(2 + \int_{x}^{x+T} T\omega_a^2 \,\mathrm{d}a + \int_{x}^{x+T}\int_{a}^{x+T} (T-(b-a))(b-a)\omega_b^2\omega_a^2 \,\mathrm{d}b\,\mathrm{d}a \\ + \int_{x}^{x+T}\int_{a}^{x+T}\int_{b}^{x+T} (T-(c-a))(c-b)(b-a)\omega_c^2\omega_b^2\omega_a^2 \,\mathrm{d}c\,\mathrm{d}b\,\mathrm{d}a + \ldots\Bigg) \end{aligned} \tag{35}$$

where $\omega_\tau = \sqrt{\gamma^2 + f(\tau)}$

This, combined with the condition discovered earlier, allows the stability domains to be found for any equation of the form of (33).

5.2 Damped Equations

Using these results, the stability domains of both the damped square wave approximation and the damped Mathieu Equation can be found. Simply transforming ω_τ to $\sqrt{\gamma^2 + f(\tau)}$ and multiplying by $e^{\gamma T}$ gives the damped stability condition from the undamped stability condition.

For the square wave equation, the undamped stability condition, (8), becomes

$$\left| e^{2\pi\gamma} \left(2c_1c_2 + \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right) \right) \right| < \begin{cases} 2, & \text{if } \gamma \geq 0 \\ 1 + e^{2\pi\gamma}, & \text{if } \gamma < 0 \end{cases}$$

Or, equivalently,

$$\left| 2c_1c_2 + \left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1} \right) \right| < \begin{cases} 2e^{-2\pi\gamma}, & \text{if } \gamma \geq 0 \\ 1 + e^{-2\pi\gamma}, & \text{if } \gamma < 0 \end{cases} \tag{36}$$

where

$$c_i = \cosh(\omega_i), \qquad s_i = \sinh(\omega_i),$$

$$\omega_1 = \sqrt{\gamma^2 + \beta + \alpha}, \quad \omega_2 = \sqrt{\gamma^2 + \beta - \alpha}$$

Expanding the linearized equation, (31) to account for friction is no harder. The only required step is multiplying by $e^{2\pi\gamma}$ and changing $f(\tau) = \beta - \alpha\cos\tau$ to $f(\tau) = \gamma^2 + \beta - \alpha\cos\tau$. Thus, any approximations made before can be made again, but multiplied by $e^{2\pi\gamma}$ and inputing, instead of β, $\gamma^2 + \beta$. So the series approximation and Taylor approximation both still apply, with the listed changes.

And so the previously mentioned ability to discover the stability of any linear, frictionless, periodically nonautonomous second order differential equation has been expanded to the ability to discover the stability of any linear, periodically nonautonomous second order differential equation with autonomous friction. Still a specific class of ODEs, but now a little more general.

6 Numerics

6.1 Comparison of Stability in the Undamped Equations

The square wave approximation is easy, and gives an intuition as to the form of stability in the true problem, but does not give the actual limits of stability for the original inverted pendulum. The solution to the linearized cosine equation, while difficult, gives stability domains that are much closer. However, the stability domains are *not* exact. Because of linearization, θ must remain near zero for the linearized equation to accurately reflect the true equation. It does not matter how close to zero $\theta(0)$ is set, (as long as $\theta(0) \neq 0$) $\theta(\tau)$ will still vary a certain amount, depending on α and β. If α and β are fairly small, this deviance will be small enough to be neglected. If α or β are too large, however, it cannot be ignored, and the linearized equation can no longer be trusted. Since nonlinear differential equations can only rarely be solved analytically, the analysis of the true, nonlinear, nonautonomous ODE must be left to numerics.

First graphs of stable and unstable equations in the phase plane (θ vs. θ') for the three ODEs, the square wave equation, the linear equation, and the nonlinear equation, top, middle, and bottom, respectively. It should be noted that for all numerically solved ODEs, the initial conditions are $\theta = 0.005$, $\theta' = 0$. The unstable are on the left, and the stable on the right. (Evaluated to $\tau = 64\pi$.)

Obviously, the stable equations stay near the origin in all cases, in fact, within 0.005 radians. This will almost always be the case for the two linear equations, and will usually be the case for the nonlinear equation. If the linear equations ever deviate more than the given initial value from the origin, they must return to below that value by the end of one period, or they are unstable; due to properties of linear equations, the next period would see a similar increase, and so on, so that the position would diverge to infinity or negative infinity. The nonlinear equation is not subject to this, but it will behave similarly to the linearized equation near the origin, so if it gets very far from the origin, in general, it will continue to diverge as well.

It is interesting to see the similarities and the differences between these graphs. On the stable side, other than the very distinct corners on the square wave equation, all three look strikingly similar. In fact, the nonlinear and linearized equations are almost imperceptively different. This supports the assumption that the linearized and nonlinear equations act in similar fashion when near the origin. On the unstable side, the square wave and linear equations act similarly, except once again for the corners, but the nonlinear equation is radically different. Not only is the behavior much more chaotic and asymmetrical, but also the scales are completely different. While the linearized and square wave equations both diverge at least to the order of 10^6, the nonlinear equation stays within the order of 10^1 in θ and less in θ'. This reflects the more physical constraints of the pendulum, as well as the chaos of nonlinear differential equations.

Of course, the most important result is stability and instability in the $\alpha - \beta$ plane. To find if the equation is stable for a given α and β, one must test to see if its solution gets far from $\theta = 0$. I chose some angle θ_0, chose initial conditions θ_a, θ'_a, numerically calculated the value of θ at T, $2T, ...,$ $8T$, and tested at each time if $\theta < \theta_0$, using Matlab. If so, then the equation is stable, and if not, then it is unstable. First, a check on the accuracy of the exact solution to the square wave is shown in Figure 7.

As can be seen, there is little, if any, difference between the two graphs. The differences take shape in how long the program takes to create the graph, and in how easily it catches the farther, smaller tongues of stability. The exact program takes much less time and catches tongues more easily.

Next, the graphs of the three equations are compared in Figure 8.

In modeling the nonlinear equation for larger α and β, neither of the other two do an accurate job. However, for smaller α and β, the linearized equation does provide a good estimate of stability, while the square wave equation becomes unstable for α smaller than the other two. When α and β are large, the linearization of the nonlinear equation is no

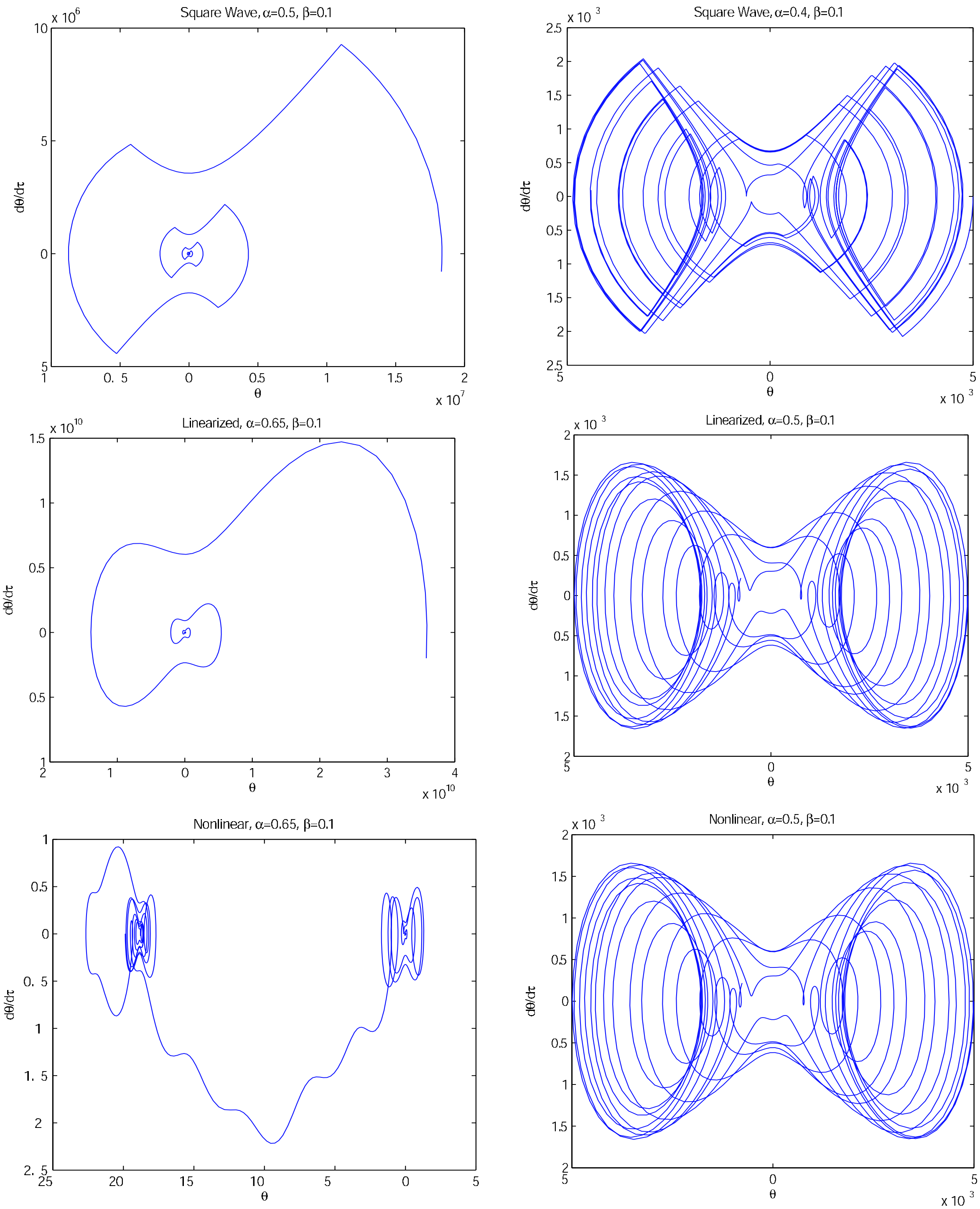


Figure 5: Stable & Unstable Equations

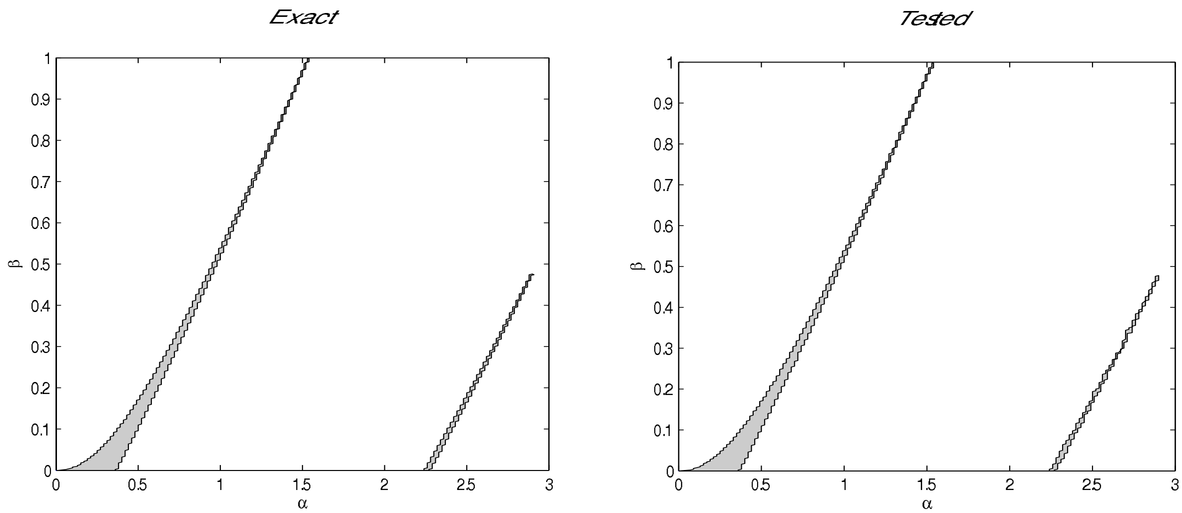


Figure 6: Square Wave: Exact vs. Test

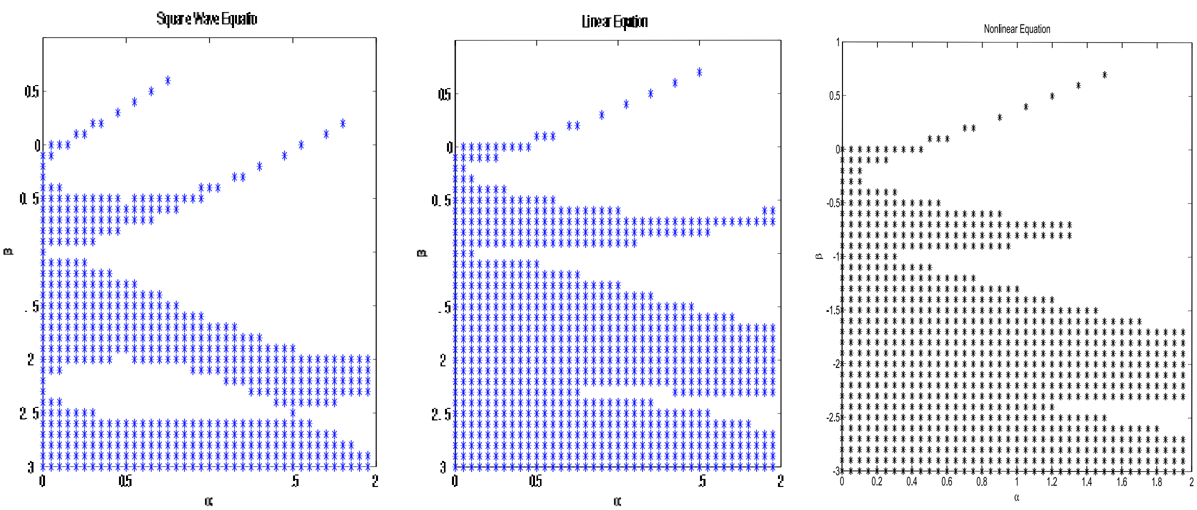


Figure 7: Comparison of Stability

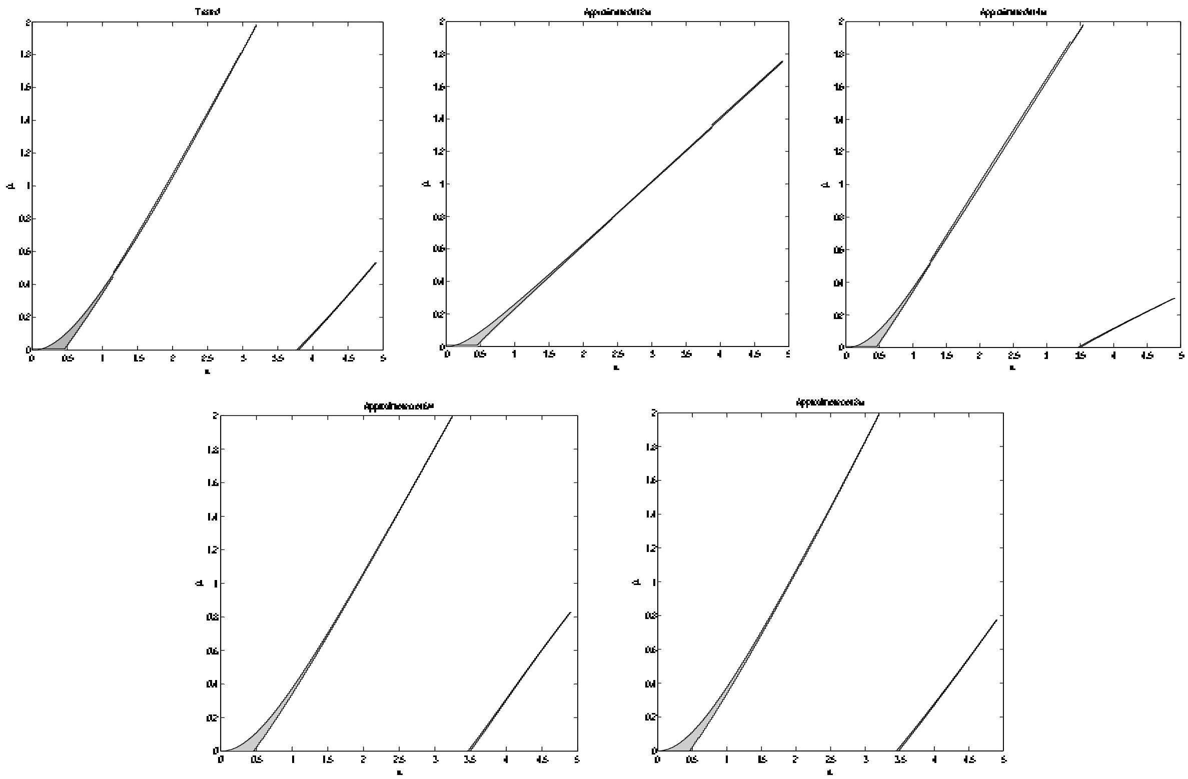


Figure 8: Linearized: Test vs. Analytical Approximations

longer viable.

The linearized and square wave equations are stable or unstable at a certain α and β regardless of the choice of initial conditions as long as those conditions are not zero, due to properties of linear equations. However, the nonlinear equation is dependant on this choice. This has a visible effect (even making the case $\alpha < 0$ nontrivial,) but luckily aslong as $\theta(0)$ and $\theta'(0)$ are near zero, the stability of the equation does not change too much.

6.2 The Linearized Equation

The nonlinear equation is the true problem, but due to a lack of ability to deal with nonlinear ODEs, the linearized equation is of as much or more fascination.

It is interesting to see how the approximations yielded by (11) reflect the true stability of the ODE. Figure 9 compares the numerically tested stability of Mathieu's Equation to approximations yielded by (11) up to $\alpha^{n-x}\beta^x$, where $n = 2,\ 4,\ 6,\ 8$.

$n = 2$ gives a very rough approximation, good for small α and β, but useless otherwise. $n = 4$ gives the first tongue almost exactly, and detects the second tongue, but one can see that the second tongue is not even on the correct spot on the α axis, and that it is misshapen. $n = 6$ and above gives the first tongue and improves accuracy even more.

These are good approximations for relatively small α and β, but the expressions, even for the first accurate approximation, 4ω, are large and clumsy. Smaller expressions and larger domains of accuracy result from using the Taylor approximations, (**??**).

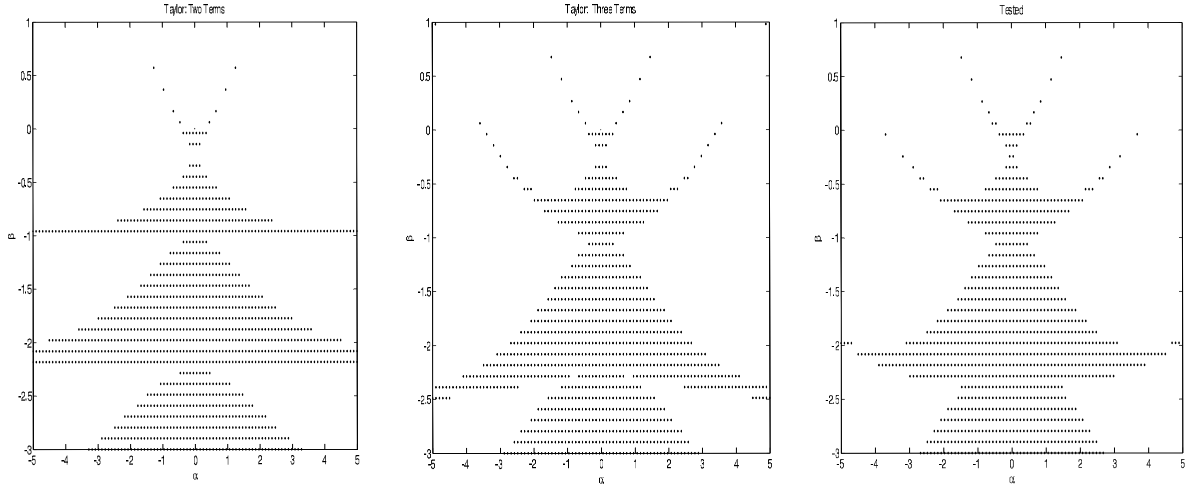


Figure 9: Linearized: Test vs. Taylor Approximations

Error lies in α, and cannot be ignored. Still, the error with only three terms of the Taylor Series is surprisingly small compared with the error in the previous approximation. Since it is valid up to terms of α^4 and all β, maybe this shouldn't be so surprising.

6.3 Damped Equations

The damped equations are interesting because they more accurately model real systems, with an appropriate choice of γ. To see the effects of friction, γ will be chosen so that it is negative and small. A change in γ has a large effect on the stability domains, so it will be sufficient to change γ by 0.05 and see the effects. The stability of the three equations with $\gamma = -0.05$ are now compared.

For the linearized and square wave equations, the effect of friction is similar. The tongues of stability are widened, though only slightly. The main effect is that the sharpness with which the instability touches the β axis is smoothed out into curves. For the nonlinear equation, friction acts simply to broaden the range of stability.

Also, it is interesting to compare the analytical derivation of the linearized equation (Taylor Approximation) and square wave equation stability to the numerically computed stability. Once again, $\gamma = -0.05$.

As can be seen, even in the square wave graphs, the two differ slightly. This difference is not due to a mistake in the analytical computation, but is actually an error on the *numerical* side. The program designed to numerically find stability was unable to find correctly the stability in a reasonable amount of time. The analytical program, at least with the square wave equation, is the more accurate.

The difference in the cosine wave graphs is partially due to error in the numerically computed graph, but most of the difference is due to the error of the Taylor approximation, especially at points further from the α axis.

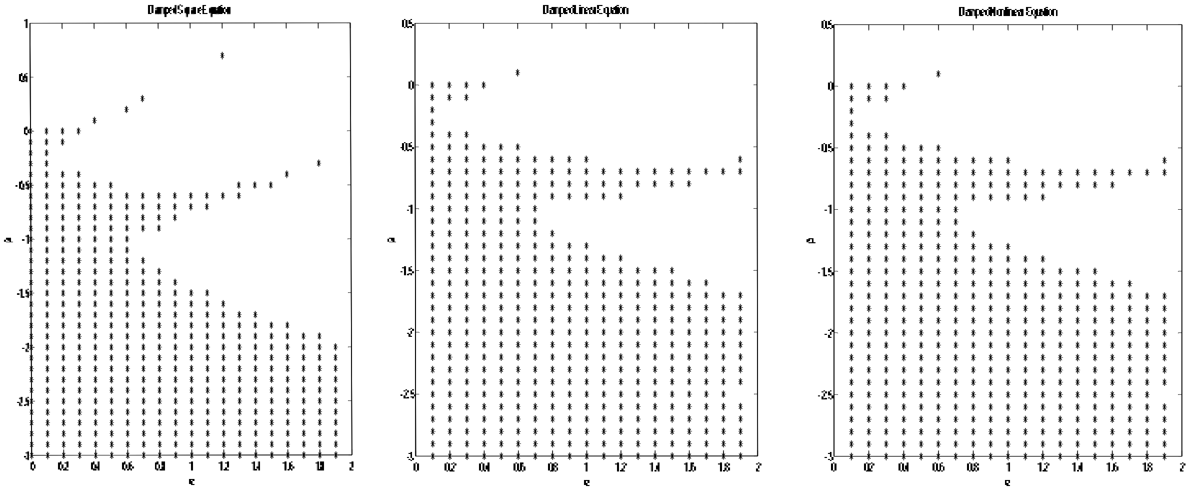


Figure 10: Stability of Damped Equations

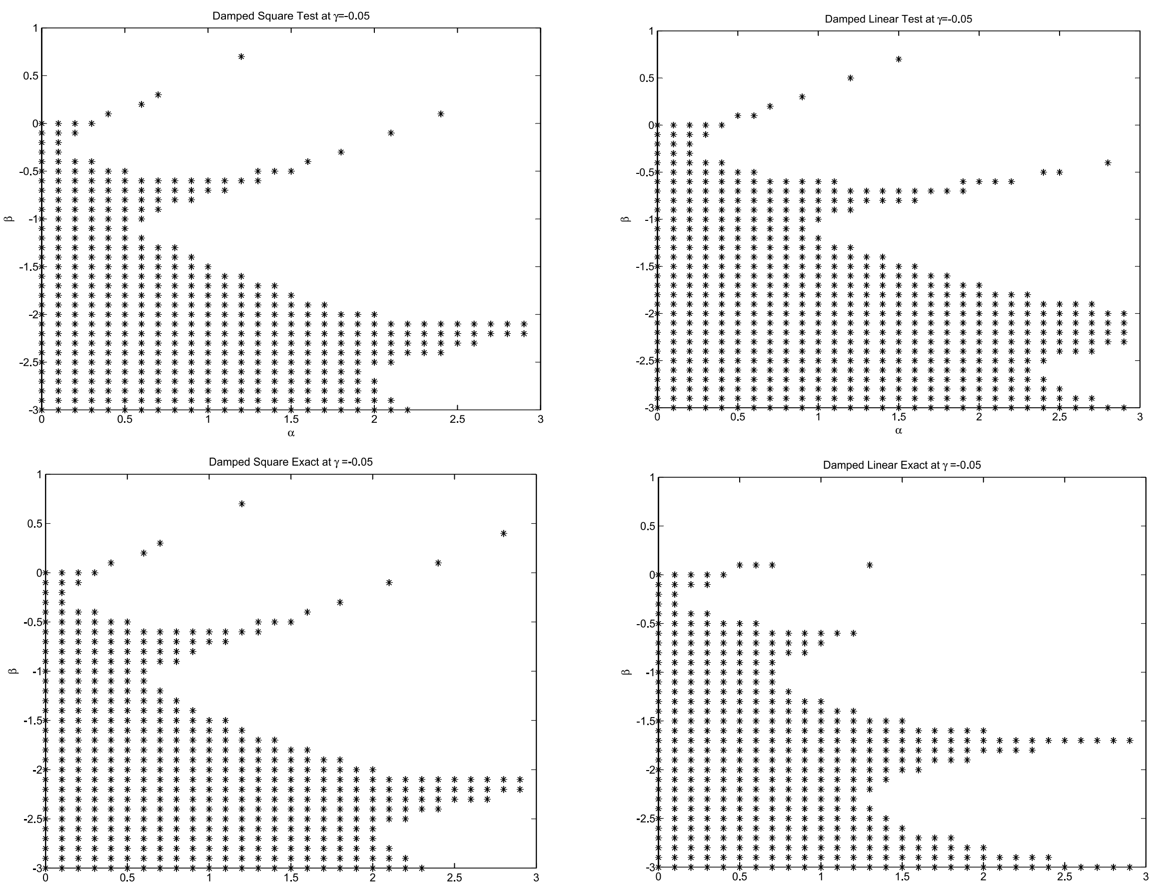


Figure 11: Comparison, Analytical to Numerical, Damped

7 Thanks

I'd like to thank Dr. Nikola Petrov for my introduction to this consuming problem and the methods of analysis. Linear algebra, stability, phase space, linearization, as well as most of the programming I now know thanks to him.

Stability domains and the equations graphed in phase space were made using MatLab. The more qualitative pictures used Adobe Illustrator. Figure 2 used both.

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