Phase Locking and Dimensions Related to Highly Critical Circle Maps

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Abstract

This article explores some basic concepts of circle maps, and presents numerical data on the dimension of unlocked sets in the parameter spaces of critical circle maps. Specifically, sine-based highly critical circle maps. Comparisons will be made to results from others' papers on similar topics.

1 Introduction

This paper and the work behind it are the result of the REU program at the University of Michigan. I will try to share what I have learned working with Professor Petrov and the results and methods of some computations that we have done.

2 Circle Maps

The basic objects of study in this paper will be circle maps. Circle maps are defined for our purposes as bijective, continuous, and orientation preserving homeomorphisms from S^1, the circle of unit circumference, onto itself. It is often convenient to talk about the *lift* of a circle map to the real line. A lift of a circle map f is a (bijective and continuous) map $F : \mathbb{R} \to \mathbb{R}$ such that there exists a covering map $\pi : \mathbb{R} \to S^1$ such that $f \circ \pi = \pi \circ F$. For our purposes, we will generally consider the projection map π to be the simple quotient map by the integers, so if, for example f were a rotation by $\pi/2$ radians, then $F(x) = x + \frac{1}{4}$. Because we are interested in iterations of these maps, we will use the notation $f^2(x) = (f \circ f)(x)$.

It is clear that $F-\mathrm{Id}$ is periodic since

$$F(x) \bmod 1 = F(x+k) \bmod 1 \text{ for k } \in 1 \text{ arbitrary.}$$

In addition, a point x on the circle will be referred to as periodic of period q if $f^q(x) = x$. This is equivalent to the statement $(F^q(x) - x) \in \mathbb{Z}$. A related quantity of circle maps is their *rotation number,* ρ. The rotation number is defined as the fractional part of

$$\rho_0 := \lim_{n\to\infty} \frac{F^n(x)}{n}.$$

Furthermore, it is simple to prove that the limit exists. Note that ρ_0 and therefore ρ is independent of the choice of x since

$$|F^n(x) - F^n(y)| \leq |(F^n(x) - x) - (F^n(y) - y)| + |x - y| \leq 1 + |x - y|$$

since $F^n(x) - x$ is periodic with period 1 since F^n is a lift of f^n, and also the range of $F(x) - x$ is of the form $[z, z+1]$ for some $z \in \mathbb{R}$. Thus,

$$\lim_{n\to\infty} \frac{F^n(x)}{n} - \lim_{n\to\infty} \frac{F^n(y)}{n} = \lim_{n\to\infty} \frac{F^n(x) - F^n(y)}{n} \leq \lim_{n\to\infty} \frac{1 + |x - y|}{n} = 0.$$

So, remembering what we were showing, ρ_0 and ρ are both independent of the choice of x. This makes periodic points extremely useful for computing the rotation number of a map. For example, if x is a periodic point of order 15, and in 15 iterations of f the point x rotates around the circle 4 times. In this case, the rotation number of f is $\frac{4}{15}$, and x is a fixed point of f^{15}. This method avoids the proper definition with lifts, but for all maps there is one lift that is exceedingly convenient, namely, the lift with the image of $[0, 1)$ contained in $[0, 2)$. This lift is convenient, since $\rho_0 = \rho$, and if for some x it happens that $(F^n(x) - x) = k$ for $k \in \mathbb{Z}$ then $\rho = \frac{k}{n}$. In general, however. it is necessary to account for the possible integer difference added on each iteration by the lift. Also, fixed points of the circle map f^q are period q points of f.

3 Phase locking, Arnol'd tongues, and the Devil's staircase

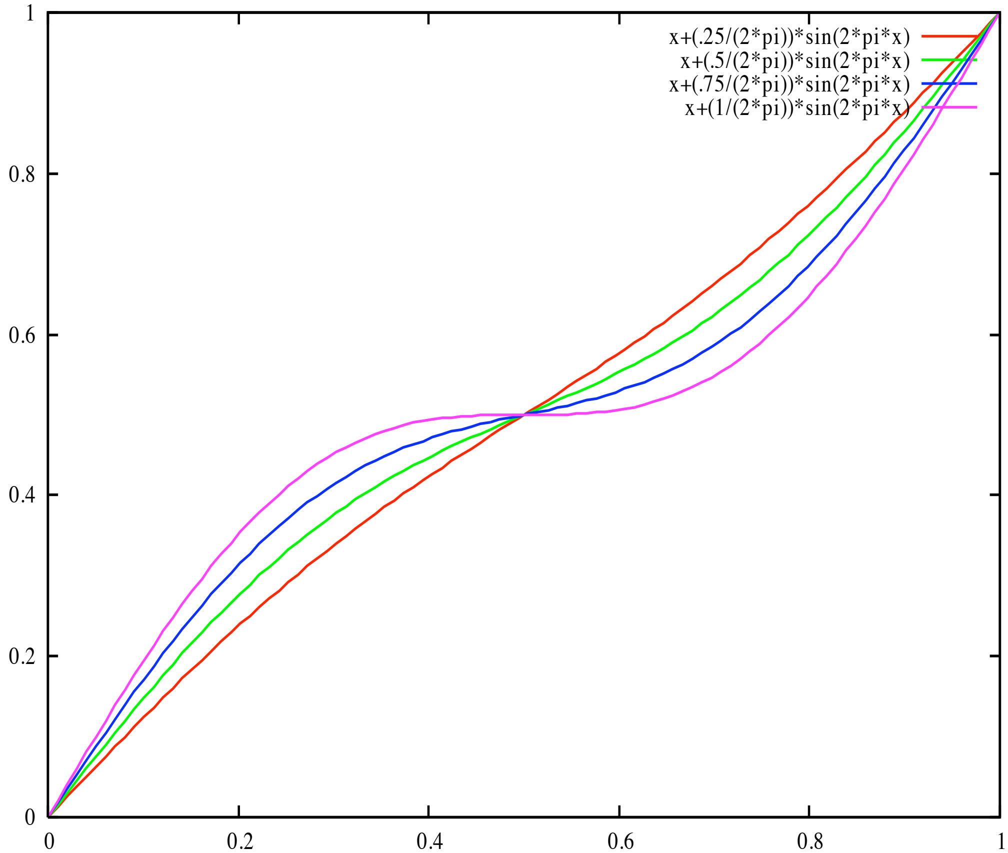

Figure 1: Graph of Arnol'd maps for $\alpha = 0$ and $\beta = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. Note that the map for $\beta = 1$ is a critical map of order 3 since it has a zero derivative at $\frac{1}{2}$, so the Taylor expansion around $\frac{1}{2}$ has no linear or quadratic term.

3.1 Arnol'd Maps

Arnol'd maps are a family of circle maps with lifts of the following form for $\alpha, \beta \in [0, 1)$:

$$F_{\alpha,\beta}(x) = x + \alpha + \frac{\beta}{2\pi}\sin(2\pi x)$$

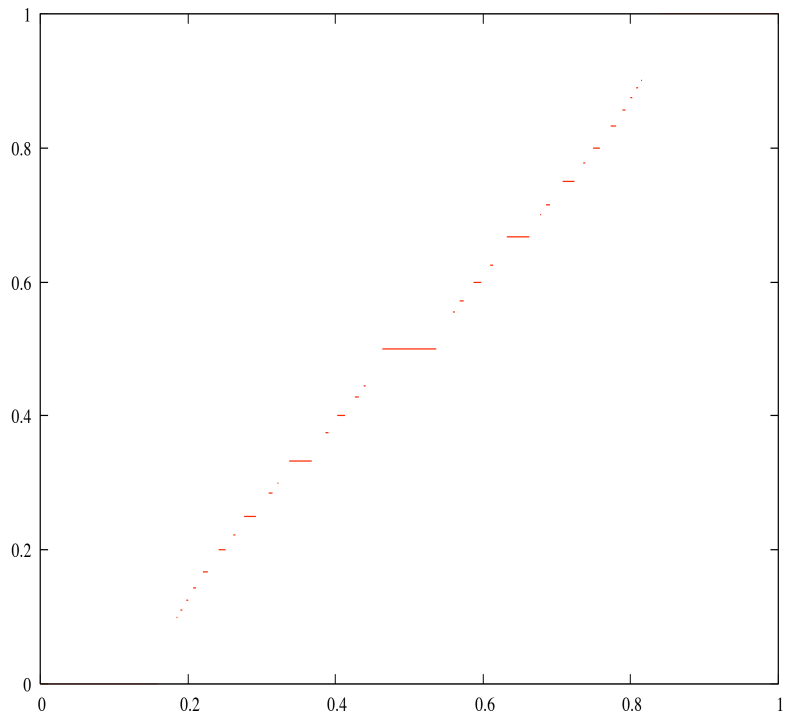

Figure 2: The Devil's Staircase for $\beta = 1$, the phase locking intervals at $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$ are quite clear. There are also large intervals for 0 and 1.

They will be considered here because they are simple and exhibit most of the interesting properties of circle maps. The behavior of the rotation number of these maps in relation to variations of α is interesting. Holding β constant, for all rational numbers $r \in [0, 1]$ there exist an interval $I_r \subset [0, 1)$ s.t. for all $\alpha \in I_r$ the rotation number $\rho_\alpha = r$. The β will be left out of the notation for simplicity.

3.2 Phase Locking

This phenomenon of the rotation number being unaffected by small changes to the parameters of the map is called *Phase Locking*. Phase locking produces a very distinctive graph of the rotation number against the parameter value that is referred to the *Devil's Staircase*. It earned the name by possessing the unpleasant quality of being locally constant wherever the rotation number is a rational number, which produces a distinctive shape.

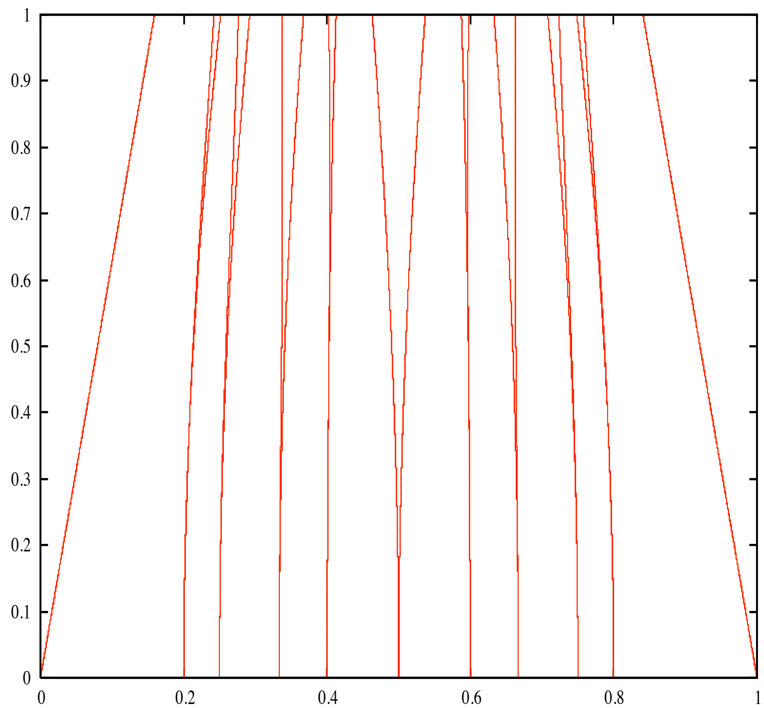

Figure 3: Graph of Arnol'd tongues. It is easy to see how the locking intervals grow from nothing when $\beta = 0$ on the x axis to $\beta = 1$. The parameter α is on the horizontal axis.

Phase locking, in addition to the devil's staircase, creates another fun looking graph with a silly name. Namely, Arnol'd tongues. This is another way of illustrating the concept of phase locking, except that it clearly shows that the length of the locking intervals grow as β increases. As β approaches 1, the total length of the locking intervals for all rationals between 0 and 1 approaches 1.

3.3 Critical Points

Indeed, it has been proposed by Lanford [6] based on numeric observation and proven rigorously by Świątek [4] that at $\beta = 1$ the total length of the locking intervals is 1. It is not, however, the case that the union of the locking intervals is the interval $(0, 1)$. At $\beta = 1$ the Arnol'd family develops a *critical* *point* where the first derivative is zero at some point and the function ceases to be diffeomorphic, but remains a homeomorphism. For $\beta > 1$ it is in fact no longer a homeomorphism. We are interested in the behavior at $\beta = 1$ since this specific case is a prototype for what happens more generally when a circle map ceases to be diffeomorphic.

Thanks to Świątek [7], we know that it is true more generally that for families of circle maps with critical points $f_t(x) = f(x) + t$ they will exhibit phase locking and that the devils staircase will also have a total length of 1, but the Hausdorff dimension of the region for which values of t contained therein result in irrational rotation numbers is bounded away from 0 by $\frac{1}{3}$. Most of my work was in this area, looking at the dimension of the complement of the Devil's Staircase as the order of the critical point increases.

4 Numerical methods for computing the phase locking intervals

4.1 General method

In order to find the dimension of the complements of the Devil's Staircases,

$$\mathcal{C} := [0,1)\backslash(\bigcup_{r\in([0,1]\bigcap\mathbb{Q})} I_r)$$

it is desirable to compute the widths of an exceedingly large number of phase locking intervals (I used some 85,000 in the cubic critical case). The computations were done to 28 digits of precision on an Apple Powerbook with a 1.25 Gigaherz G4. Up to denominators of 530, 296, 271, 246, 220, and 200 were computed for the degree 3, 5, 7, 9 , 11, 13 cases, respectively. For the curious, the computations took approximately 84 hours total, with much of that being the 11 and 13 degree critical cases. The algorithm for finding the width of the locking interval makes use of two subroutines written by Forsythe, Malcom, and Moler [3], namely `zeroin` and `fmin` (referred to as `fmin2` in my code, since the C standard libraries now contain a function already called `fmin`).

The mechanism by which we found the widths of the phase locking interval is to start with the rotation number for which we are trying to find the width of the phase locking interval. The rotation number is defined by the numerator p and denominator q. Then we set up a function called `ns` that returns $f_\alpha^q(x) - x - p$, so that zeros occur at the periodic points of period q where f_α^q rotates x p times around the circle. The lower endpoint of the phase locking interval is then the value of α where the minimum of `ns` is zero, and the upper endpoint is the value of α for which the maximum of `ns` is zero. Because we have a convenient function to return the minimum of functions, the maximum is computed by computing the minimum of the negative of `ns`, which is called `mns` as it is *minus* `ns`.

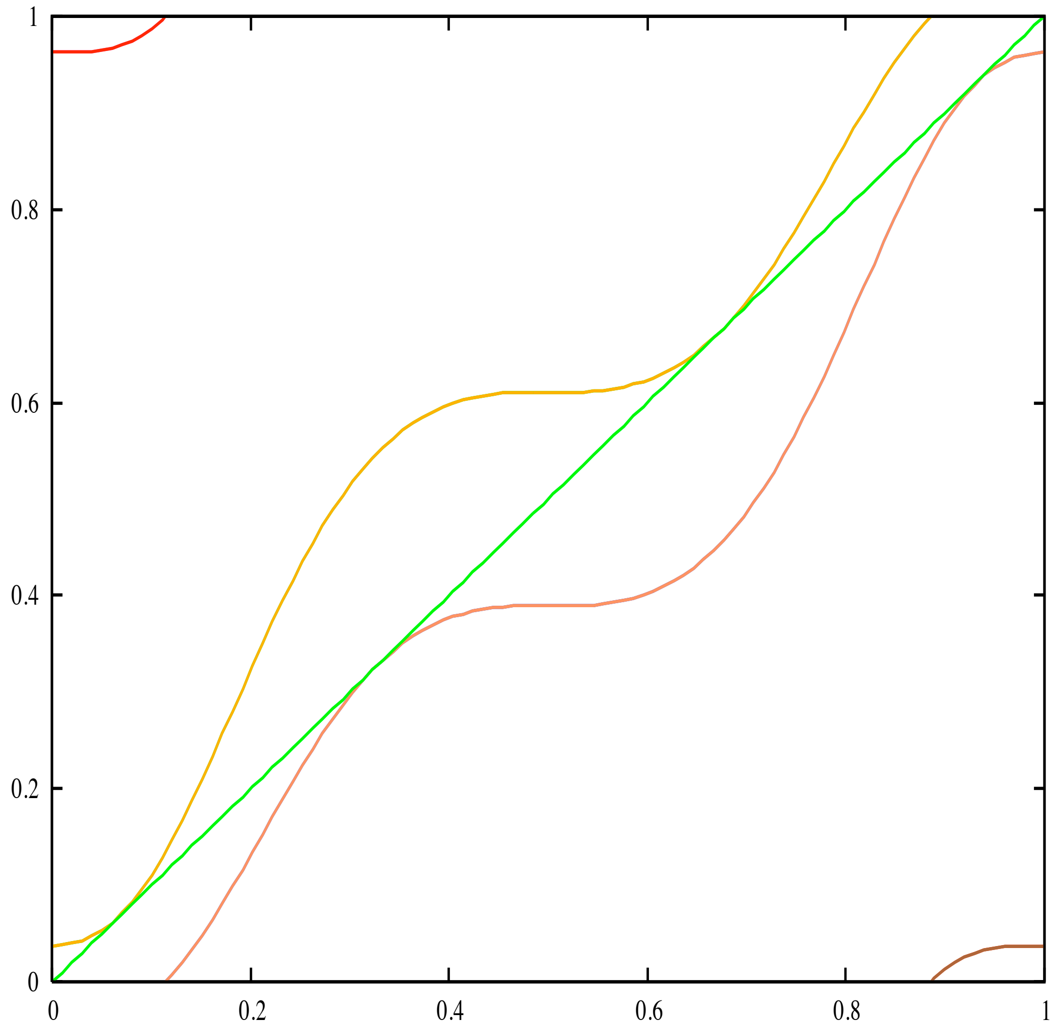

Figure 4: The graphs of the critical Arnol'd map for the minimal and maximal values of α for which there is a fixed point of f_α^2. This illustrates the method used to find the phase locking intervals, since these are the Arnol'd maps corresponding to the two endpoints of the locking interval for $\frac{1}{2}$. It is clear that anywhere outside this interval, f_α^2 would not intersect the graph of the identity function, i.e. the diagonal. There would therefore be no points of period 2, so the function would not have rotation number $\frac{1}{2}$

4.2 Implementation of the method

```

DBL width_locking( DBL beta, DBL num, DBL den)
{
  beta_global = beta;
  num_global  = num;
  den_global  = den;

  /*  zeroin( zero, one, ming, tol_zeroin)  is the right end
      of the phase-locking interval for rotation number p/q
      for this value of beta, and
      zeroin( zero, one, maxg, tol_zeroin)  is the left end
      of the same interval
  */
  return( zeroin( zero, one, ming, tol_zeroin) \
          - zeroin( zero, one, maxg, tol_zeroin) );
}/************************************************************/

```

This is the function that is called first in order to find the length of the phase locking interval. It uses `zeroin` to find a zero between zero and one (they are merely appropriately named variables, since it is easier to use variables when maintaining high precision is important) of `ming` and `maxg` respectively. These functions, `ming` and `maxg`, are functions of α.

```
DBL ming( DBL x)
{
  alpha_global = x;
  return ( ns( fmin2( zero, one, ns, tol_fmin) ) );
}/*************************************************************/

DBL maxg( DBL x)
{
  alpha_global = x;
  return ( mns( fmin2( zero, one, mns, tol_fmin) ) );
}/*************************************************************/
```

Here we see `ming` and `maxg` in all of their glory. They simply return the minimum value of `ns` or `mns` respectively for $x \in [0, 1]$.

```
DBL ns (DBL x)
{
  int i;
  DBL xin = x, x0 = x;
  for ( i = 0 ; i < den_global ; i++ )    x0 = ones(x0);
  return (x0 - 1.0*num_global - xin);
}/*************************************************************/

DBL mns (DBL x)
{
  int i;
  DBL xin = x, x0 = x;
  for ( i = 0 ; i < den_global ; i++ )    x0 = ones(x0);
  return (1.0*num_global + xin - x0);
}/*************************************************************/
```

These are the `ns` and `mns` functions that iterate the function and subtract off the numerator of the rotation number that is being checked. They both call the cryptically named `ones` function, which is simply the Arnol'd function which also depends upon the globally defined α and β. This means that this function has a minimum, or maximum, at zero only when α is at the lower, or upper end of the phase locking interval.

```
for ( den = 1; den < (n +1) ; den ++)
{
        for (num  = 1 ; num < den ; num++)
        {
                k = 0;

                for ( i = 1; i <= num && k != 1 ; i++)
                {
                        if ( (floorl(num/i) == num/i)  && \
                 (floorl(den/i) == den/i) && i != 1  )
                        {
                                k = 1;
                        }
                }
                if ( k == 0 )
                {
                        printf( "%.1Lf %.1Lf ", num, den);
                        t = width_locking( beta, num, den);
                        printf ( "%.40Lf\n", t);
                        s+= t;
                }
        }
}
```

4.3 Comments

In order to find all of the rotation numbers for which to find the phase locking intervals, ideally it would be possible to find all the phase locking intervals above a certain length. Unfortunately, there is no known way to do this, so we fall back on simply computing them for all denominators below some value. There are many efficient ways to do this, but there is also a phenomenally inefficient but very easy way. The inefficient way consists of counting up through denominators an starting at a numerator of 1 on every one and checking every numerator between 1 and the denominator to see if the fraction is unique. This can be seen in the section of code above. I've chosen the inefficient way for two reasons. First, the amount of time the computer spends in this part of the program is miniscule. For each phase locking interval the computer minimizes a function that contains a sine function iterated up to 500 times to 30 digits of precision.

I actually wrote a much better version of this part of the program, using Farey series (A Farey series consists of all the rationals, that are between 0 and 1 with denominator less than some value, in ascending order) that should have sped up this part approximately 20 times. There was, however, no noticeable difference in speed between the two methods up to a denominator of 50. I assume that at some point the difference would have been noticeable, but probably at a much larger denominator.

The other reason I went with the brute force method is that it leaves you with useable data if you stop it early. Using Farey series, an early stop would leave you with denominators up to the limit you had set, only up to some point between 0 and 1. An early stop of the brute force method, on the other hand, merely reduces the precision. Also, in order to obtain more data with the brute force method, you can start up where you left off, instead of having to recompute everything you did before.

There were, of course, several other programs, for determining the dimension and such. They however, were rather exceptionally uninteresting, and so will not be included here.

5 Results on the dimension of the *unlocked* set in parameter space

5.1 Box Dimension

In order to computer compute the dimension of $\mathcal{C}$ we first need to define what we mean by dimension. We will use the simple notion of box dimension. Box dimension is based on the idea of covering a set with n-dimensional cubes of side length ϵ. $N_\epsilon(S)$ is defined as the minimum number of such n-dimensional cubes needed to cover the set S. The dimension is then defined as the number d s.t. $\lim_{\epsilon\to 0} \frac{N_\epsilon(S)}{\epsilon^d} = k$ for some constant k. In the event that d exists, we use the more useful definition

$$d = \lim_{\epsilon\to 0} \frac{\log k - \log N_\epsilon(S)}{\log \epsilon} = -\lim_{\epsilon\to 0} \frac{\log N_\epsilon(S)}{\log \epsilon}.$$

The method I used to compute the dimension of $\mathcal{C}$ is the same as that used by Jensen, Bak, and Bohr [5]. It yields the box dimension, which could be considered a tad simplistic. But it is certainly useful and has the distinct advantage of being easy to compute. It can be computed by summing the total length of all the phase locking intervals with lengths longer than r,

$$S(r) = \sum_{p,q \ s.t. \ I_{\frac{p}{q}} > r} I_{\frac{p}{q}}.$$

Then $N_r(\mathcal{C}) = \frac{(1-S(r))}{r}$. The box dimension of the $\mathcal{C}$ is

$$\lim_{r \to 0} \left(\frac{\log N_r(\mathcal{C})}{(\log \frac{1}{r})} \right).$$

Then, if we graph $\log N_r(\mathcal{C})$ against $\log \frac{1}{r}$ and they form a straight line, the slope would correspond to the box dimension.

For example, consider the standard middle third Cantor set, $\mathcal{T}$. In order to compute the dimension, we'll consider r to be powers of $\frac{1}{3}$, so

$$\lim_{r \to 0} \frac{\log N_r(\mathcal{T})}{(\log \frac{1}{r})} = \lim_{n \to \infty} \frac{\log N_{3^n}(\mathcal{T})}{(\log 3^n)} = \lim_{n \to \infty} \frac{\log \frac{(1-S(\frac{1}{3^n}))}{r}}{(n \log 3)}$$

where $S(\frac{1}{3^n})$ is the total length of intervals in the complement of the Cantor set in the unit interval, or

$$S(\frac{1}{3^n}) = \sum_{m=1}^{n} \frac{2^{n-1}}{3^n} = \frac{1}{3} \left(\frac{1}{1-\frac{2}{3}} - \frac{\frac{2^n}{3^{n+1}}}{1-\frac{2}{3}} \right) = 1 - \frac{2^n}{3^{n+1}}.$$

So, substituting back into $\lim_{n \to \infty} \frac{\log \frac{(1-S(3^n))}{r}}{(n \log 3)}$ we get

$$\lim_{n \to \infty} \frac{\log \frac{(\frac{2^n}{3^{n+1}})}{\frac{1}{3^n}}}{(n \log 3)} = \lim_{n \to \infty} \frac{n \log 2 - (n+1) \log 3 + n \log 3}{(n \log 3)} = \frac{\log 2}{\log 3}$$

This method of computing agrees with Hausdorf on the simple middle third Cantor set. Delbourgo and Kenny [1] have also computed dimensions for similar circle maps, and they arrived at slightly different values. This could, hopefully, but not likely, be because of the fact that I computed far more locking intervals, or due to the specific maps used. The maps used should not affect the results, however, since maps of the same criticality always have the same associated dimension. Jensen [5] used the same method

Criticality	*My Dimension*	*Dimension from Delbourgo and Kenny [1]*
3	$0.868 \pm .001$	$.871 \pm .001$
5	$0.815 \pm .003$	$0.821 \pm .001$
7	$0.791 \pm .002$	$0.795 \pm .002$
9	$0.771 \pm .002$	n/a
10	n/a	0.770 ± 0.002
11	$0.765 \pm .002$	n/a
13	$0.745 \pm .001$	n/a

as I did for computing the dimension, and he got a dimension of 0.8700 for the cubic critical case, closer to Delbourgo and Kenny's than to mine.

Delbourgo and Kenny studied the family of maps

$$f(\theta) = \Omega + \frac{\theta|2\theta|^{z-1}}{2}, -\frac{1}{2} < \theta < \frac{1}{2},$$

which has the added benefit of being of being able to evaluate critical maps of arbitrary criticality (they give dimensions for 1.1, 1.5, 1.8, and 2.5 in their paper). They do, however, have the disadvantage of not being analytic like functions based on sine maps like the ones that I considered.

What is really interesting though, is the graph of the dimension against the degree of criticality. The first 5 show a reasonable quasi-exponential decay rate, but the order 13 criticality departs from this. It would be interesting to see whether this is really the case, and how the dimensions of further degrees of criticality behave.

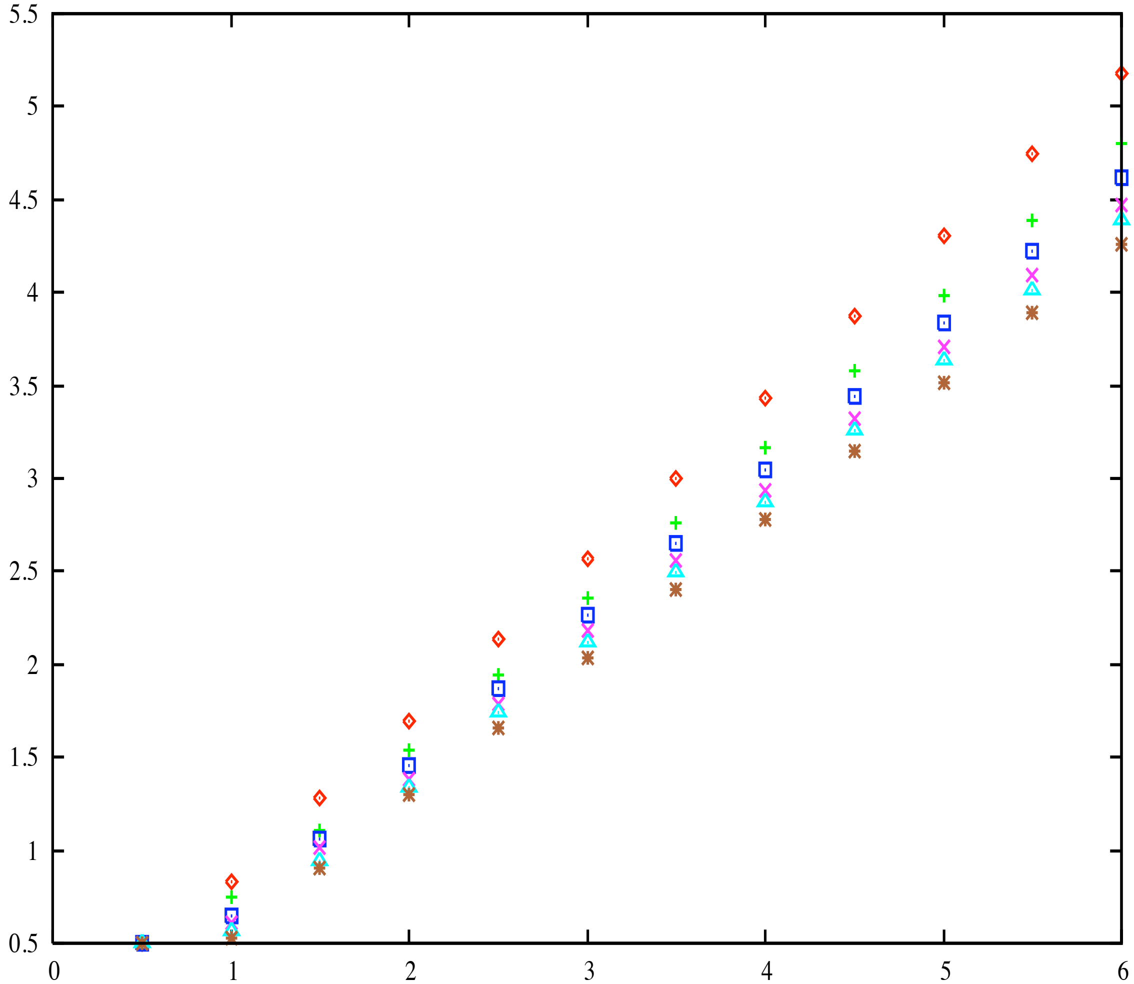

Figure 5: Plot of $\log N(r)$ vs. $\log \frac{1}{r}$. The slope (dimension of $\mathcal{C}$) decreases with increasing severity of the criticality of the map.

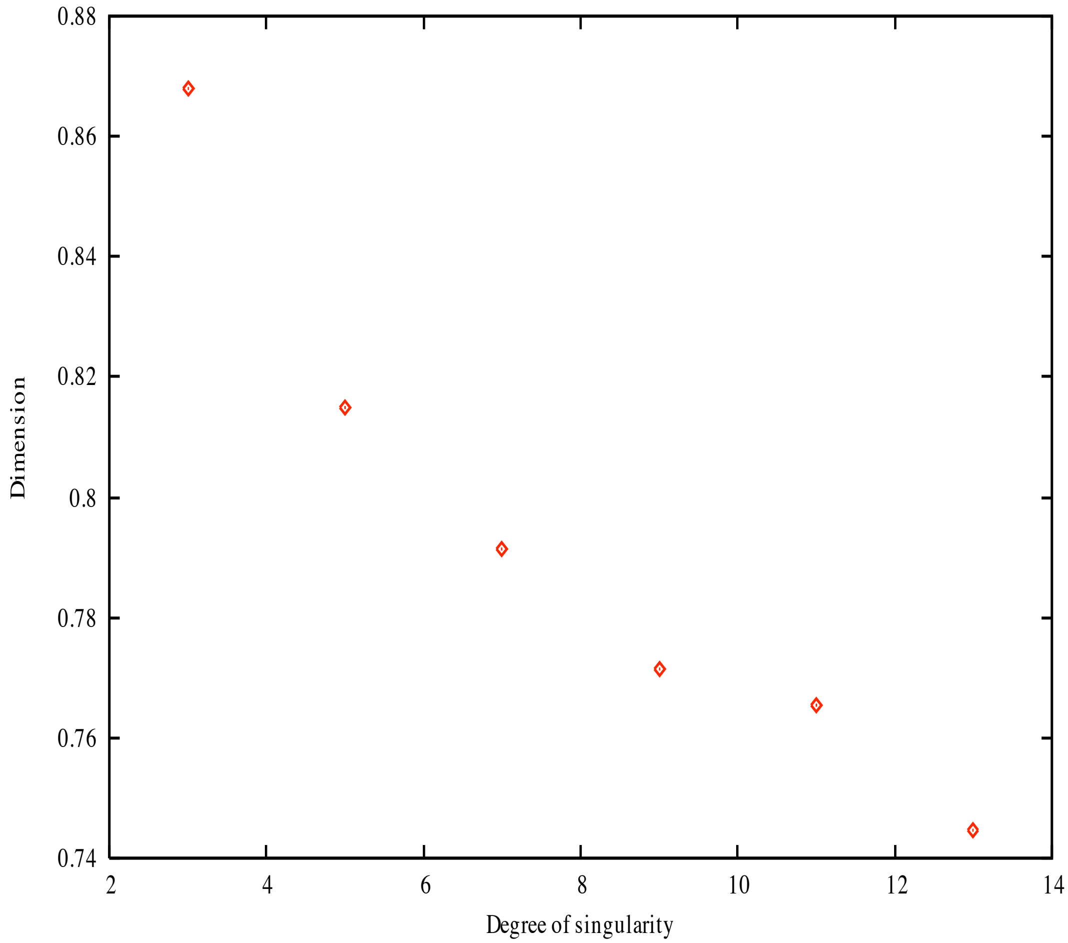

Figure 6: This shows the dimension of $\mathcal{C}$ of the maps corresponding to the degree of criticality of the map.

A Appendix

$$S_k := \sin(k\pi x)$$

quintic critical map – for $\frac{4}{5} < K < \frac{8}{5}$

$$f^{\mathrm{S}}(x) = x + \omega - \frac{1}{2\pi}\left(KS_2 + \frac{9-8K}{10}S_4 + \frac{3K-4}{15}S_6\right)$$

septimic critical map – for $\frac{4}{5} < K < \frac{8}{5}$

$$f^{\mathrm{S}}(x) = x + \omega - \frac{1}{2\pi}\left(KS_2 + \frac{6-5K}{5}S_4 + \frac{45K-64}{105}S_6 + \frac{3-2K}{28}S_8\right)$$
$$\approx \frac{32\pi^6(8-5K)}{35}x^7 + O(x^9)$$

nonic critical map – for $1 < K < \frac{5}{3}$

$$f^{\mathrm{D}}(x) = x + \omega - \frac{1}{2\pi}\left(KS_2 + \frac{10-8K}{7}S_4 + \frac{27K-40}{42}S_6 + \frac{25-16K}{84}S_8 + \frac{5K-8}{210}S_{10}\right)$$
$$\approx \frac{128\pi^8(5-3K)}{63}x^9 + O(x^{11})$$

order-11 critical map – for $\frac{8}{7} < K < \frac{12}{7}$

$$f^{\mathrm{E}}(x) = x + \omega - \frac{1}{2\pi}\left(KS_2 + \frac{45-35K}{28}S_4 + \frac{105K-160}{126}S_6 + \frac{45-28K}{84}S_8 + \frac{1925K-3168}{25410}S_{10} + \frac{5-3K}{396}S_{12}\right)$$
$$\approx \frac{256\pi^{10}(12-7K)}{231}x^{11} + O(x^{13})$$

order-13 critical map – for $\frac{5}{4} < K < \frac{7}{4}$

$$f^{\mathrm{T}}(x) = x + \omega - \frac{1}{2\pi}\left(KS_2 + \frac{21-16K}{12}S_4 + \frac{9K-14}{9}S_6 + \frac{105-64K}{132}S_8 + \frac{25K-42}{165}S_{10} + \frac{245-144K}{5148}S_{12} + \frac{7K-12}{3003}S_{14}\right)$$
$$\approx \frac{1024\pi^{12}(7-4K)}{429}x^{13} + O(x^{15})$$

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