Stationary distributions of semistochastic processes with disturbances at random times and with random severity

Maria Conceição A. Leite, Nikola P. Petrov

Department of Mathematics, University of Oklahoma, 601 Elm Avenue, Norman, OK 73019, USA

Ensheng Weng

Department of Botany and Microbiology, University of Oklahoma, 770 Van Vleet Oval, Norman, OK 73019, USA

Abstract

We consider a semistochastic continuous-time continuous-state space random process that undergoes downward disturbances with random severity occurring at random times. Between two consecutive disturbances the evolution is deterministic, given by an autonomous ordinary differential equation. The times of occurrence of the disturbances are distributed according to a general renewal process. At each disturbance the process gets multiplied by a continuous random variable ("severity") supported on $[0, 1)$. The inter-disturbance time intervals and the severities are assumed to be independent random variables that also do not depend on the history.

We derive an explicit expression for the conditional density connecting two consecutive post-disturbance levels, and an integral equation for the stationary distribution of the post-disturbance levels. We obtain an explicit expression for the stationary distribution of the random process. Several concrete examples are considered to illustrate the methods for solving the integral equations that occur.

Keywords: Semistochastic process, Disturbances with random severity, Renewal process, Catastrophe, Disaster

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1. Introduction and set-up of the problem

Random disturbances of physical, chemical and biological systems occur commonly. The effects of such phenomena have been studied intensively in population dynamics. A problem that motivated this paper was related to the carbon content of an ecosystem – recently some authors have identified disturbances (extreme droughts, fires, insect outbreaks, etc.) as key forces driving the dynamics of carbon (Thornton *et al* [1], Pregitzer and Euskirchen [2], Bond-Lamberty *et al* [3], Running [4]), and, more generally, as a factor in the dynamics of vegetation (see, e.g., Clark [5], and the recent papers D'Odorico *et al* [6] and Beckage *et al* [7]).

Another situation in which disturbances play an important role is the stochastic phenotype switching in microbial populations in response to sudden catastrophic events in the environment (see, e.g., Kussell and Leibler [8], Wolf *et al* [9], or, for a more mathematical exposition, Gander *et al* [10]). Visco *et al* [11] recently suggested a mathematical model of the dynamics of such systems that is somewhat reminiscent of the model we consider in this paper. We, however, will use the carbon content of an ecosystem as a motivating example throughout the paper because its nature matches more closely our assumptions in the main theorems of our paper.

The amount of carbon in the ecosystem increases due to photosynthesis, and after a long time approaches the corresponding carrying capacity of the ecosystem. Occasionally, however, a forest fire, an extreme drought, or some other process occurring on much shorter time scale than the normal growth of plants destroys some part of the ecosystem. We call such a fast process of decimation of the forest a *disturbance*, and consider the disturbances as instantaneous events. Therefore, we arrive at the following continuous-time continuous-state space stochastic process $X(t)$ modeling the carbon mass in the ecosystem. In the time between two consecutive disturbances $X(t)$ evolves deterministically, governed by the autonomous ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = g(X(t)) \ . \tag{1}$$

Sometimes processes like X are called *semistochastic* in the literature. We will need that the solution of the differential equation (1) be an invertible function, so we impose the condition that the function g be strictly positive; it is allowed to be zero only at the ends of the interval where $X(t)$ is allowed to vary (as in the cases (6) and (7) below). In the context of the carbon content problem, to account for the constant growth rate of the plants and the saturation effects (due to finite carrying capacity), one can take, for example, $g(x) = 1 - x$ (cf. Equation (6) below).

We assume that $X(t)$ is non-negative, which is the case in many applications. The quantity $X(t)$ changes with a downward jump at some discrete set of random times $\Theta_1 < \Theta_2 < \Theta_3 < \cdots$. The random times Θ_j form a renewal process. If the cause of the disturbance is natural, one can perhaps assume that Θ_j's come from a Poisson process, but one can, for example, consider the case of controlled forest fires performed at scheduled times (unless a natural fire occurs), in which case the process will not be Poisson. We assume that the fraction of the forest that is destroyed in the disturbance – termed the *severity* of the disturbance – is a continuous random variable with a known distribution supported on $[0, 1)$. To simplify our considerations, we assume that the times of occurrence of the disturbances do not depend on the state of the system (i.e., on $X(t)$).

To formulate the questions we study in this paper, we need to introduce some notation. We define the pre-disturbance levels Y_n^- and the post-disturbance levels Y_n of the process by

$$Y_n^- := \lim_{t\uparrow\Theta_n} X(t), \qquad Y_n := \lim_{t\downarrow\Theta_n} X(t) \ ; \tag{2}$$

as represented pictorially in Figure 1. Let the severity of the nth disturbance be determined

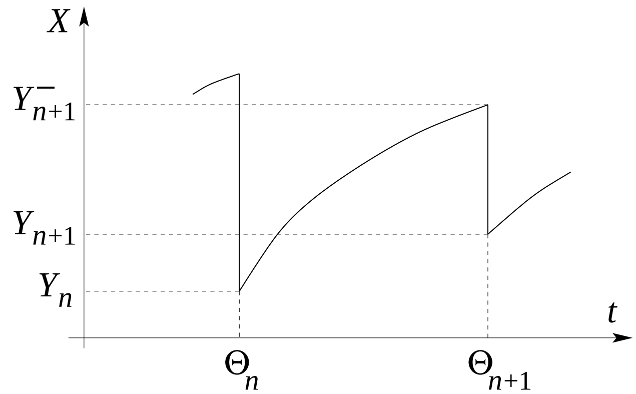

Figure 1: On the definition of the Θ_n, Y_n^-, and Y_n (2). The process between Θ_n and Θ_{n+1} is governed by (1).

by a continuous random variable U_n relating the pre- and post-disturbance levels:

$$Y_n = U_n \, Y_n^- \ . \tag{3}$$

Clearly, U_n should be supported on the interval $[0, 1)$, and it is reasonable to assume that these random variables are independent and identically distributed. We also want that U_n be independent of the process $X(t)$. Of course, the practical measurements of the distributions of the inter-disturbance times and the severity of the disturbances in practical situations is a complicated issue (see, e.g., Reed *et al* [12]).

One meaningful question that can be asked is to find the distributions of the pre- and post-disturbance levels. Another interesting problem is to find the fraction of time the process X spends in the long run in a certain measurable set $A \subset \mathbb{R}$. We will assume that the long-time distribution of X can be described by a p.d.f. f_X defined as

$$\int_A f_X(x) \, \mathrm{d}x = \lim_{T \to \infty} \frac{1}{T} \, \mu\big(\{t \in [0, T] : X(t) \in A\}\big) \ , \tag{4}$$

for any Borel set A (where μ stands for the Lebesgue measure), whenever this limit exists.

We will call p.d.f.'s like f_X *stationary* or *invariant* distributions, and the measures they define *invariant* measures.

In this paper we derive an explicit expression for the conditional probability density function $f_{Y_{n+1}|Y_n}$ relating two consecutive post-disturbance levels, in terms of the p.d.f. f_T of the inter-disturbance times T_n, the p.d.f. f_U of the severities U_n, and the function g in (1). This function is the kernel in an integral equation for the stationary probability density function f_Y of the post-disturbance levels Y_n. To solve the integral equation for f_Y, we transform it to a differential equation that is easier to solve. We also derive an explicit expression for f_X (4) in terms of f_Y, f_U, and g.

We use our theoretical results to compute explicitly f_Y and f_X in several particular cases (in each case we also give the allowed range of the initial condition x_0):

(A) the constant growth rate equation:

$$g(x) = 1 \ , \qquad x_0 \in [0, \infty) \ ; \tag{5}$$

(B) an equation corresponding to growth with saturation (modeling the carbon mass in an ecosystem):

$$g(x) = 1 - x \ , \qquad x_0 \in [0, 1) \ ; \tag{6}$$

(C) the logistic equation:

$$g(x) = x(1 - x) \ , \qquad x_0 \in (0, 1) \ ; \tag{7}$$

(D) the exponential growth equation:

$$g(x) = x \ , \qquad x_0 \in (0, \infty) \ . \tag{8}$$

In each of these examples we will assume that the disturbance times Θ_n form a Poisson process of rate λ (thus, the inter-disturbance times $T_n = \Theta_n - \Theta_{n-1}$ are exponentially distributed) and that U is uniformly distributed on the interval $[0, 1)$.

Note that in (7) and (8) we required that the initial condition x_0 be strictly positive – the reason for this is that in these cases $g(0) = 0$, so that if the population is zero at some moment, it will be always zero after that.

We emphasize that, generally, the post-disturbance levels Y_n do not form a Markov chain. A notable exceptional case when Y_n do form a Markov chain is when all inter-disturbance times T_n are equal to some constant (independent of n). Another particular case is when the T_n's are random but all disturbances bring the population to zero (i.e, if $\mathbb{P}(U = 0) = 1$; this kind of disturbance is usually called a *disaster* or a *catastrophe*) – then the random variables Y_n are independent.

Since our exposition is directed mostly towards applied scientists, we ignore the complicated question of deriving general conditions for existence of stationary distributions, and instead try to construct the stationary distributions in several concrete examples. (For a physically meaningful situation where such a question is solved see, e.g., Doering and Horsthemke [13].) Since our construction of invariant p.d.f.'s is explicit, the uniqueness of the stationary p.d.f. – whenever such p.d.f. exists – follows from the uniqueness of the solution of the differential equation obtained in the process of computing $f_{Y_{n+1}|Y_n}$. (See, however, Remark 3).

Our results are exact and purely analytical, so they can be used as a testing tool of numerical simulations of random processes.

Note that ideas similar to the ones considered in this paper can be applied, *mutatis mutandis*, to the case of a non-positive function g in (1) and disturbances at the times Θ_n that go upwards – a generalization of the so-called "counter model" [14, Sec. 7.3] in which the sizes of the upward jumps $(Y_n - Y_n^-)$ are random variables independent of the pre-disturbance levels Y_n^- (while in our formalism they depend on Y_n^-).

Below we mention some works that study similar problems, but under different assumptions. Usually in them the system is modeled by a continuous-time, discrete state space random process.

In an early work on impulsively forced population models, Kaplan *et al* [15] studied a branching process with disasters occurring at the times of an independent renewal process. The population is represented by a discrete state space process, and each particle alive at the time of the disaster survives the disaster with certain probability (independently on the other particles).

Bartoszyński [16] proposed a model of development of rabies in a human organism. In his model, the population of viruses is represented by a discrete state space random process that jumps upwards by jumps that are independent identically distributed random variables with a given distribution, while between the jumps the population decreases exponentially. The times of occurrence of upward jumps are allowed to be population-dependent.

Hanson and Tuckwell [17] studied extinction times in logistically growing populations that undergo disasters of fixed magnitude that occur in time as a Poisson process. In their subsequent work [18], they analyzed disasters that are a constant multiple of the current population size, and in [19] they considered more general distributions of the size of the disasters. These and other models were discussed by Lande [20] in the context of risks of population extinction from different factors. Cairns [21] considered general methods for numerical solutions for first-exit times in semistochastic processes.

Pakes *et al* [22] considered a continuous state space model of a population that grows exponentially between a discrete set of times at which a random fraction of the population emigrates; in particular, the authors studied the extinction time. They assumed that the sizes of the groups emigrating are independent identically distributed random variables with a given distribution, while the rate of occurrence of the events of emigration depends on the population size. (This model is a generalization of the one developed by Bartoszyński [16].) Brockwell *et al* [23] considered a discrete state space birth-immigration-catastrophe process, in which the population can undergo two types of jumps at random times – upward jumps by one and downward jumps of sizes given by certain particular types of probability distributions. In another paper [24], the same authors studied catastrophe processes to a stochastic process with continuous state space. Brockwell [25, 26] generalized the considerations of [23] to a wider class of probability distributions of the downward jumps. More recently, discrete state space processes with different types of catastrophes are considered by Cairns and Pollett [27], Pollett *et al* [28], Economou and Gómez-Corral [29], among others.

One particular type of catastrophes in stochastic population models with discrete state space – namely, binomial catastrophes – seems to be particularly similar in spirit to our assumption that $X(t)$ is multiplied by a random number in $[0, 1)$ at each occurrence of a disturbance. In such models the disturbances occur according to a renewal process (usually a Poisson process), and each individual survives after the disturbance with probability $p \in [0, 1)$, independently of the other individuals. Therefore, the expected population size right after the disturbance is equal to the pre-disturbance population multiplied by p. Such processes are studied by Brockwell *et al* [23], Economou [30], and Artalejo *et al* [31].

We would also like to note that the problem studied in this paper is somewhat similar to the so-called "processes with random regulation" (see, e.g., [32, 33, 34] and the references therein).

The plan of our paper is the following: in Section 2 we derive an integral equation for the conditional p.d.f. $f_{Y_{n+1}|Y_n}$ and an expression for the stationary distribution f_Y of the post-disaster levels, in Section 3 we derive a formula for the stationary distribution f_X of the process X, in Section 4 we apply the general theory to the concrete examples (5)–(8), Section 5 is devoted to a comparison (on a "physical level of rigor") of a discrete stochastic process with its continuous analogue.

2. Deriving an expression for the conditional p.d.f. $f_{Y_{n+1}|Y_n}$ of two consecutive post-disturbance levels and an integral equation for the stationary p.d.f. f_Y of Y_n

We start by introducing some notations related to the solutions of the differential equation (1). Let $\phi_t(x_0)$ be the flow of (1) with initial condition $x(0) = x_0$ in absence of disturbances in the time interval $[0, t)$. Let $\psi(x_0, x_1)$ stand for the duration of time that the system needs to evolve from x_0 to x_1 in absence of disturbances, i.e.,

$$x_1 = \phi_t(x_0) \quad \Longleftrightarrow \quad t = \psi(x_0, x_1) \ . \tag{9}$$

Since g is strictly positive (except possibly vanishing at the ends of allowed interval for its argument), $\phi_t(x_0)$ is a strictly increasing function of t, and the function ψ is well-defined. From (9) it is clear that $\psi(x_0, x_1) < 0$ for $x_1 < x_0$.

Throughout the paper, we denote by d the maximum allowed value that $X(t)$ can take, i.e., $d = 1$ in the cases (6) and (7), while $d = \infty$ in the cases (8) and (5). We always assume that the initial value x_0 is in the interval $[0, d)$ or $(0, d)$.

Theorem 1. *Let X be a continuous-time continuous-state space semistochastic process satisfying the following assumptions:*

(a) *$X(t)$ takes values in the interval $[0, d)$;*

(b) *the disturbances occur at times $\Theta_1 < \Theta_2 < \Theta_3 < \cdots$, and the inter-disturbance times $T_n = \Theta_n - \Theta_{n-1}$ (positive by definition) are independent identically distributed continuous random variables which are also independent of the process X; f_T stands for the p.d.f. of the common distribution of the T_n's;*

(c) *between two consecutive disturbances $X(t)$ evolves deterministically governed by the autonomous differential equation (1); the function g in (1) is non-negative (except possibly vanishing at the ends of interval where $X(t)$ can take values) and Lipschitz (so that (1) has a unique solution);*

(d) the random variables U_n (3) connecting the pre- and post-disturbance levels (2) – and, hence, determining the severity of the disturbances – are independent identically distributed random variables that do not depend on the process X; the common p.d.f. of the U_n's is f_U, which is supported on $[0, 1)$.

Then the conditional p.d.f. of the $(n+1)$st post-disturbance level Y_{n+1} conditioned on the nth post-disturbance level Y_n is given by

$$f_{Y_{n+1}|Y_n}(x_{n+1}|x_n) = \int_{\max\{x_n, x_{n+1}\}}^{d} \frac{f_T(\psi(x_n, x_{n+1}^-))}{g(x_{n+1}^-)} \, f_U\left(\frac{x_{n+1}}{x_{n+1}^-}\right) \frac{\mathrm{d}x_{n+1}^-}{x_{n+1}^-} \ . \tag{10}$$

If the post-disturbance levels Y_n tend asymptotically (as $n \to \infty$) to some continuous random variable Y, then p.d.f. f_Y of Y satisfies the integral equation

$$f_Y(y) = \int_0^d f_{Y_{n+1}|Y_n}(y|x) \, f_Y(x) \, \mathrm{d}x \ , \tag{11}$$

as well as the non-negativity and normalization conditions: $f_Y \geq 0$ and $\int_0^d f_Y(x)\,\mathrm{d}x = 1$.

Proof. Assume that Δx_{n+1}^- is an infinitesimal positive increment and ignore terms of order higher than linear in it. Then for $x_{n+1}^- \in [x_n, d)$ and $x_{n+1}^- + \Delta x_{n+1}^- \in [x_n, d)$, we have

$$\begin{aligned}
f_{Y_{n+1}^-|Y_n}(x_{n+1}^-|x_n)\,\Delta x_{n+1}^- &= \mathbb{P}\big(Y_{n+1}^- \in (x_{n+1}^-, x_{n+1}^- + \Delta x_{n+1}^-]\big|Y_n = x_n\big) \\
&= \mathbb{P}\big(T_{n+1} \in (\psi(x_n, x_{n+1}^-), \psi(x_n, x_{n+1}^- + \Delta x_{n+1}^-)]\big|Y_n = x_n\big) \\
&= \mathbb{P}\big(T_{n+1} \in (\psi(x_n, x_{n+1}^-), \psi(x_n, x_{n+1}^- + \Delta x_{n+1}^-)]\big) \\
&= \mathbb{P}\left(T_{n+1} \in \left(\psi(x_n, x_{n+1}^-), \psi(x_n, x_{n+1}^-) + \frac{\partial \psi}{\partial x_{n+1}^-}(x_n, x_{n+1}^-)\,\Delta x_{n+1}^-\right]\right) \\
&= \mathbb{P}\left(T_{n+1} \in \left(\psi(x_n, x_{n+1}^-), \psi(x_n, x_{n+1}^-) + \frac{\Delta x_{n+1}^-}{g(x_{n+1}^-)}\right]\right) \\
&= \frac{f_T(\psi(x_n, x_{n+1}^-))}{g(x_{n+1}^-)}\,\Delta x_{n+1}^- \ ,
\end{aligned}$$

therefore

$$f_{Y_{n+1}^-|Y_n}(x_{n+1}^-|x_n) = \frac{f_T(\psi(x_n, x_{n+1}^-))}{g(x_{n+1}^-)} \, \chi_{[x_n, d)}(x_{n+1}^-) \ . \tag{12}$$

The expression for the conditional p.d.f. $f_{Y_{n+1}|Y_{n+1}^-}$ follows directly from the definition:

$$\begin{aligned}
f_{Y_{n+1}|Y_{n+1}^-}(x_{n+1}|x_{n+1}^-)\,\Delta x_{n+1} &= \mathbb{P}\big(Y_{n+1} \in (x_{n+1}, x_{n+1} + \Delta x_{n+1}]\big|Y_{n+1}^- = x_{n+1}^-\big) \\
&= \mathbb{P}\big(U_{n+1} Y_{n+1}^- \in (x_{n+1}, x_{n+1} + \Delta x_{n+1}]\big|Y_{n+1}^- = x_{n+1}^-\big) \\
&= \mathbb{P}\big(U_{n+1} x_{n+1}^- \in (x_{n+1}, x_{n+1} + \Delta x_{n+1}]\big) \\
&= \mathbb{P}\left(U_{n+1} \in \left(\frac{x_{n+1}}{x_{n+1}^-}, \frac{x_{n+1} + \Delta x_{n+1}}{x_{n+1}^-}\right]\right) \\
&= f_U\left(\frac{x_{n+1}}{x_{n+1}^-}\right) \frac{\Delta x_{n+1}}{x_{n+1}^-} \ ,
\end{aligned}$$

hence

$$f_{Y_{n+1}|Y^-_{n+1}}(x_{n+1}|x^-_{n+1}) = \frac{1}{x^-_{n+1}}\, f_U\left(\frac{x_{n+1}}{x^-_{n+1}}\right) \ . \tag{13}$$

We use (13) and (12) to obtain the desired conditional p.d.f. $f_{Y_{n+1}|Y_n}$. Recall that $\operatorname{supp} f_T = (0,\infty)$, $\operatorname{supp} f_U = [0,1)$, and $\psi(x,y) \geq 0$ for $0 \leq x \leq y < d$, to obtain

$$\begin{aligned} f_{Y_{n+1}|Y_n}(x_{n+1}|x_n) &= \int f_{Y_{n+1}|Y^-_{n+1}}(x_{n+1}|x^-_{n+1})\, f_{Y^-_{n+1}|Y_n}(x^-_{n+1}|x_n)\, \mathrm{d}x^-_{n+1} \\ &= \int_{\max\{x_n,x_{n+1}\}}^{d} \frac{f_T(\psi(x_n,x^-_{n+1}))}{g(x^-_{n+1})}\, f_U\left(\frac{x_{n+1}}{x^-_{n+1}}\right) \frac{\mathrm{d}x^-_{n+1}}{x^-_{n+1}} \ . \end{aligned}$$

Finally, the integral equation (11) for the stationary p.d.f. of the post-disturbance level follows directly from

$$f_{Y_{n+1}}(x_{n+1}) = \int_0^d f_{Y_{n+1}|Y_n}(x_{n+1}|x_n)\, f_{Y_n}(x_n)\, \mathrm{d}x_n \ .$$

Since (11) is linear in f_Y, we need to impose the normalization condition in order to find the overall multiplicative constant in f_Y. □

It is instructive to check that $\int_0^d f_{Y_{n+1}|Y_n}(x_{n+1}|x_n)\, \mathrm{d}x_{n+1} = 1$. We change the order of integration and merge two integrals into one to obtain

$$\begin{aligned} \int_0^d f_{Y_{n+1}|Y_n}(x_{n+1}|x_n)\, \mathrm{d}x_{n+1} &= \int_0^d \mathrm{d}x_{n+1} \int_{\max\{x_n,x_{n+1}\}}^{d} \frac{\mathrm{d}x^-_{n+1}}{x^-_{n+1}} \frac{f_T(\psi(x_n,x^-_{n+1}))}{g(x^-_{n+1})}\, f_U\left(\frac{x_{n+1}}{x^-_{n+1}}\right) \\ &= \int_0^{x_n} \mathrm{d}x_{n+1} \int_{x_n}^{d} \frac{\mathrm{d}x^-_{n+1}}{x^-_{n+1}} \frac{f_T(\psi(x_n,x^-_{n+1}))}{g(x^-_{n+1})}\, f_U\left(\frac{x_{n+1}}{x^-_{n+1}}\right) \\ &\quad + \int_{x_n}^{d} \mathrm{d}x_{n+1} \int_{x_{n+1}}^{d} \frac{\mathrm{d}x^-_{n+1}}{x^-_{n+1}} \frac{f_T(\psi(x_n,x^-_{n+1}))}{g(x^-_{n+1})}\, f_U\left(\frac{x_{n+1}}{x^-_{n+1}}\right) \\ &= \int_{x_n}^{d} \mathrm{d}x^-_{n+1} \frac{f_T(\psi(x_n,x^-_{n+1}))}{g(x^-_{n+1})} \int_0^{x_n} \frac{\mathrm{d}x_{n+1}}{x^-_{n+1}} f_U\left(\frac{x_{n+1}}{x^-_{n+1}}\right) \\ &\quad + \int_{x_n}^{d} \mathrm{d}x^-_{n+1} \frac{f_T(\psi(x_n,x^-_{n+1}))}{g(x^-_{n+1})} \int_{x_n}^{x^-_{n+1}} \frac{\mathrm{d}x_{n+1}}{x^-_{n+1}} f_U\left(\frac{x_{n+1}}{x^-_{n+1}}\right) \\ &= \int_{x_n}^{d} \mathrm{d}x^-_{n+1} \frac{f_T(\psi(x_n,x^-_{n+1}))}{g(x^-_{n+1})} \int_0^1 \mathrm{d}\xi\, f_U\left(\xi\right) \\ &= \int_0^{\infty} \mathrm{d}t\, f_T(t) = 1 \ . \end{aligned}$$

We used the substitutions $\xi = x_{n+1}/x^-_{n+1}$ and $t = \psi(x_n, x^-_{n+1})$, and the fact that the definition (9) and the differential equation (1) yield $x^-_{n+1} = \phi_t(x_n)$, $\mathrm{d}x^-_{n+1}/\mathrm{d}t = g(x^-_{n+1})$.

3. An expression for the stationary p.d.f. f_X of the stochastic process X

Suppose that we know the stationary p.d.f. $f_Y(x)$ of the post-disturbance levels. In this section we will derive two expressions for the stationary p.d.f. of $X(t)$ defined by (4) .

Theorem 2. *Under the assumptions of Theorem 1, if the stationary p.d.f. of X exists, then it is given by the expressions*

$$\begin{aligned} f_X(x) &= \frac{\chi_{(0,d)}(x)}{g(x)} \int_0^x \mathrm{d}x_n\, f_Y(x_n) \int_{\psi(x_n,x)}^{\infty} \mathrm{d}\tau_{n+1}\, \frac{f_T(\tau_{n+1})}{\tau_{n+1}} && (14) \\ &= \frac{\chi_{(0,d)}(x)}{g(x)} \int_0^{\infty} \mathrm{d}\tau_{n+1}\, \frac{f_T(\tau_{n+1})}{\tau_{n+1}} \int_{\max\{0,\phi_{-\tau_{n+1}}(x)\}}^{x} \mathrm{d}x_n\, f_Y(x_n)\ . && (15) \end{aligned}$$

Proof. Consider the evolution of the system between the nth and the $(n+1)$st disturbances. We will first find the p.d.f. $f_{X|Y_n,T_{n+1}}$ of X conditioned on the values of the nth post-disturbance level and the time interval $T_{n+1} = \Theta_{n+1} - \Theta_n$ between the nth and $(n+1)$st disturbance. Assume that $Y_n = x_n$ and $T_{n+1} = \tau_{n+1}$. Let x^* be some value between x_n and x_{n+1}^-, and $\Delta x^* > 0$ be an infinitesimal increment, so that

$$x_n \le x^* < x^* + \Delta x^* < x_{n+1}^-\ .$$

Let τ^* and $\tau^* + \Delta\tau^*$ be the time elapsed between the moment of occurrence of the nth disturbance and the moment when the level has values x^* and $x^* + \Delta x^*$, respectively. Using (9), we can write $x^* = \phi_{\tau^*}(x_n)$ and $x^* + \Delta x^* = \phi_{\tau^*+\Delta\tau^*}(x_n)$ or, equivalently, $\tau^* = \psi(x_n, x^*)$ and $\tau^* + \Delta\tau^* = \psi(x_n, x^* + \Delta x^*)$. Clearly,

$$0 \le \tau^* < \tau^* + \Delta\tau^* < \tau_{n+1}\ .$$

The differential equation (1) implies that $\frac{\Delta x^*}{\Delta\tau^*} = \frac{\mathrm{d}x}{\mathrm{d}t}\big|_{t=\Theta_n+\tau^*} = g(x^*)$. The probability of $X(t)$ to be in the interval $(x^*, x^* + \Delta x^*]$ between times Θ_n and Θ_{n+1} is equal to the fraction of time between the nth and $(n+1)$st disturbance that the system spends in the interval $(\Theta_n + \tau^*, \Theta_n + \tau^* + \Delta\tau^*]$:

$$\begin{aligned} f_{X|Y_n,T_{n+1}}(x^*|x_n,\tau_{n+1})\,\Delta x^* &= \mathbb{P}(X(\Theta_n+\tau) \in (x^*, x^*+\Delta x^*] | \tau \in (0,\tau_{n+1}]) \\ &= \frac{\Delta\tau^*}{\tau_{n+1}} = \frac{1}{\tau_{n+1}}\,\frac{\Delta x^*}{g(x^*)}\ , \end{aligned}$$

which implies the following expression for $f_{X|Y_n,T_{n+1}}$:

$$f_{X|Y_n,T_{n+1}}(x|x_n,\tau_{n+1}) = \frac{\chi_{(x_n,\phi_{\tau_{n+1}}(x_n)]}(x)}{\tau_{n+1}\,g(x)}\ . \tag{16}$$

The p.d.f. of X is obtained from the conditional one (16) by averaging over Y_n and T_{n+1}:

$$\begin{aligned} f_X(x) &= \int \mathrm{d}\tau_{n+1}\, f_T(\tau_{n+1}) \int \mathrm{d}x_n\, f_Y(x_n)\, f_{X|Y_n,T_{n+1}}(x|x_n,\tau_{n+1}) \\ &= \int \mathrm{d}\tau_{n+1}\, f_T(\tau_{n+1}) \int \mathrm{d}x_n\, f_Y(x_n) \frac{\chi_{(x_n,\phi_{\tau_{n+1}}(x_n)]}(x)}{\tau_{n+1} g(x)}\ . \end{aligned}$$

Because of the nature of the problem, this integration should be over $\tau_{n+1} \geq 0$, $x_n \in (0, d)$. However, because of the indicator function in the integrand and since $X(t)$ is always assumed to be non-negative, for a given value of x, the domain of integration is

$$0 \leq x_n \leq x \leq \phi_{\tau_{n+1}}(x_n) \ .$$

From this (14) and (15) follow easily. □

Checking that f_X integrates to 1 can again be done by changing the order of integrations and substitutions similar to the ones in the verification of the normalization condition for $f_{Y_{n+1}|Y_n}$ after Theorem 1.

4. Particular cases of disturbances of uniform severity occurring in time as a Poisson process

In this section we apply the expressions derived in Theorems 1 and 2 to the four particular cases (A)–(D) listed in the introduction. In all cases we assume that:

(a) the times Θ_n of occurrence of disturbances form a Poisson process of rate λ, hence the inter-disturbance times $T_n = \Theta_n - \Theta_{n-1}$ are independent exponential random variables:

$$T_n \sim \mathrm{Exp}\,(\lambda) \quad \Longrightarrow \quad f_T(t) = \lambda \mathrm{e}^{-\lambda t} \chi_{[0,\infty)}(t) \tag{17}$$

(where χ_A stands for the indicator function of the set A);

(b) the severity of the disturbances are determined by the independent random variables U_n which are uniformly distributed on $[0, 1)$:

$$U_n \sim \mathrm{Unif}\,([0,1)) \quad \Longrightarrow \quad f_U(x) = \chi_{[0,1)}(x) \ . \tag{18}$$

4.1. Case (A) – constant growth rate

If the deterministic change of X is described by the constant growth rate equation (5), then the functions ϕ and ψ (9) are

$$\phi_t(x_0) = x_0 + t \ , \qquad \psi(x_0, x) = x - x_0 \ . \tag{19}$$

Substituting the functions g (5), f_T (17), f_U (18), and ψ (19) into the expression (10) for the conditional p.d.f. $f_{Y_{n+1}|Y_n}$ relating two consecutive post-disturbance levels, we obtain

$$\begin{aligned} f_{Y_{n+1}|Y_n}(x_{n+1}|x_n) &= \int_{\max\{x_n, x_{n+1}\}}^{\infty} \lambda \mathrm{e}^{-\lambda(x_{n+1}^- - x_n)} \frac{1}{x_{n+1}^-} \, \mathrm{d}x_{n+1}^- \\ &= \begin{cases} -\lambda \mathrm{e}^{\lambda x_n} \mathrm{Ei}(-\lambda x_n) & \text{if } 0 < x_{n+1} < x_n \ , \\ -\lambda \mathrm{e}^{\lambda x_n} \mathrm{Ei}(-\lambda x_{n+1}) & \text{if } 0 < x_n < x_{n+1} \ , \end{cases} \end{aligned}$$

where Ei is the exponential integral [35, Eqn. 5.1.2]. The integral equation (11) takes the form

$$f_Y(x) = -\lambda \mathrm{Ei}(-\lambda x) \int_0^x \mathrm{e}^{\lambda y} f_Y(y)\,\mathrm{d}y - \lambda \int_x^\infty \mathrm{e}^{\lambda y}\,\mathrm{Ei}(-\lambda y) f_Y(y)\,\mathrm{d}y\ , \qquad x \in [0,\infty)\ . \quad (20)$$

We can rewrite the right-hand side of (20) by using the integral representation of Ei and changing the order of integration in the second term:

$$\begin{aligned} -\lambda \int_x^\infty \mathrm{e}^{\lambda y}\,\mathrm{Ei}(-\lambda y) f_Y(y)\,\mathrm{d}y &= \lambda \int_x^\infty \mathrm{d}y\,\mathrm{e}^{\lambda y}\, f_Y(y) \int_{\lambda y}^\infty \mathrm{d}t \frac{\mathrm{e}^{-t}}{t} \\ &= \lambda \int_{\lambda x}^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-t}}{t} \int_x^{t/\lambda} \mathrm{d}y\,\mathrm{e}^{\lambda y}\, f_Y(y) \\ &= \lambda \int_{\lambda x}^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-t}}{t} \left[P\left(\tfrac{t}{\lambda}\right) - P(x)\right] \\ &= \lambda \int_{\lambda x}^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-t}}{t} P\left(\tfrac{t}{\lambda}\right) + \lambda\,\mathrm{Ei}(-\lambda x)\,P(x)\ , \end{aligned}$$

where we have set $P(z) = \int_0^z \mathrm{e}^{\lambda y}\, f_Y(y)\,\mathrm{d}y$. Substituting this back in (20), we obtain

$$f_Y(x) = \lambda \int_{\lambda x}^\infty \mathrm{d}t\,\frac{\mathrm{e}^{-t}}{t} \int_0^{t/\lambda} \mathrm{d}y\,\mathrm{e}^{\lambda y} f_Y(y)\ , \qquad x \in [0,\infty)\ . \quad (21)$$

The integral equations (20) or, equivalently, (21) can be solved by converting them to a differential equation. Differentiate (21) once, resp. twice, with respect to x to obtain

$$f_Y'(x) = -\lambda \frac{\mathrm{e}^{-\lambda x}}{x} \int_0^x \mathrm{d}y\,\mathrm{e}^{\lambda y} f_Y(y)\ , \quad (22)$$

$$f_Y''(x) = \lambda \mathrm{e}^{-\lambda x}\,\frac{\lambda x + 1}{x^2} \int_0^x \mathrm{d}y\,\mathrm{e}^{\lambda y} f_Y(y) - \frac{\lambda}{x}\, f_Y(x)\ . \quad (23)$$

Using (22) to express the integral in (23) in terms of $f_Y'(x)$ yields the following linear second-order differential equation for f_Y:

$$x f_Y''(x) + (\lambda x + 1) f_Y'(x) + \lambda f_Y(x) = 0\ . \quad (24)$$

The left-hand side of this differential equation is a total derivative [36, Eq. 2.116], so we integrate it once to obtain the first-order linear differential equation

$$x f_Y'(x) + \lambda x f_Y(x) = C_1\ .$$

In the limit $x \to 0$, the left-hand side of this equation tends zero, so $C_1 = 0$, and the equation further reduces to a separable equation with general solution $f_Y(x) = C_2 \mathrm{e}^{-\lambda x}$. (The point $x = 0$ is a singular point for the equation (24), but this does not influence our result since what we are solving is not (24) but the integral equation (21).) Using the normalization condition $\int_0^\infty f_Y(x)\,\mathrm{d}x = 1$ to find the constant C_2, we obtain that asymptotically, the post-disturbance levels are exponentially distributed:

$$f_Y(x) = \lambda \mathrm{e}^{-\lambda x} \chi_{[0,\infty)}(x) \ . \tag{25}$$

The mean and the variance of Y are $\mathbb{E}\, Y = \frac{1}{\lambda}$ and $\operatorname{Var} Y = \frac{1}{\lambda^2}$, respectively.

For computing f_X, we can use either (14) or (15) to obtain

$$f_X(x) = \lambda \left[(\gamma + \ln(\lambda x))\, \mathrm{e}^{-\lambda x} - \mathrm{Ei}(-\lambda x) \right] \chi_{[0,\infty)}(x) \ , \tag{26}$$

where $\gamma = 0.5772156649\ldots$ is the Euler's constant. The mean and the variance of X are

$$\mathbb{E}\, X = \frac{3}{2\lambda} \ , \qquad \operatorname{Var} X = \frac{17}{12\,\lambda^2} \ .$$

4.2. Case (B) – growth with saturation (modeling the carbon content of an ecosystem)

In this case equation (6) yields

$$\phi_t(x_0) = 1 + (x_0 - 1)\mathrm{e}^{-t} \ , \qquad \psi(x_0, x) = \ln \frac{1 - x_0}{1 - x} \ . \tag{27}$$

From (6), (10), (17), (18), and (27), we obtain

$$\begin{aligned} f_{Y_{n+1}|Y_n}(x_{n+1}|x_n) \;&=\; \int_{\max\{x_n, x_{n+1}\}}^{1} \lambda \left(\frac{1 - x_{n+1}^-}{1 - x_n} \right)^{\lambda} \frac{1}{x_{n+1}^-(1 - x_{n+1}^-)} \, \mathrm{d}x_{n+1}^- \\ &=\; \begin{cases} \dfrac{\lambda}{(1 - x_n)^\lambda}\, B_{1-x_n}(\lambda, 0) & \text{if } 0 < x_{n+1} < x_n \ , \\ \dfrac{\lambda}{(1 - x_n)^\lambda}\, B_{1-x_{n+1}}(\lambda, 0) & \text{if } 0 < x_n < x_{n+1} \ , \end{cases} \end{aligned}$$

where $B_z(a, b)$ is the incomplete beta function [35, Eqn. 6.6.1]. Therefore, the integral equation (11) for f_Y becomes

$$f_Y(x) = \lambda\, B_{1-x}(\lambda, 0) \int_0^x \frac{f_Y(y)}{(1 - y)^\lambda} \, \mathrm{d}y + \lambda \int_x^1 \frac{B_{1-y}(\lambda, 0)\, f_Y(y)}{(1 - y)^\lambda} \, \mathrm{d}y \ , \qquad x \in [0, 1) \ . \tag{28}$$

We can rewrite this integral equation in an equivalent form by using the integral representation of the incomplete beta function to rewrite $B_{1-y}(\lambda, 0)$ in (28), then change the order of integration (like in the derivation of (21)), arriving finally at the integral equation

$$f_Y(x) = \lambda \int_0^{1-x} \mathrm{d}t \, \frac{t^{\lambda - 1}}{1 - t} \int_0^{1-t} \mathrm{d}y \frac{f_Y(y)}{(1 - y)^\lambda} \ , \qquad x \in [0, 1) \ . \tag{29}$$

To solve (28) or, equivalently, (29), we convert them to a differential equation. Differentiate (29) once, resp. twice, with respect to x,

$$f_Y'(x) = -\lambda \frac{(1 - x)^{\lambda - 1}}{x} \int_0^x \frac{f_Y(y)}{(1 - y)^\lambda} \, \mathrm{d}y \ ,$$

$$f_Y''(x) = -\lambda \left[\frac{(1-x)^{\lambda-1}}{x}\right]' \int_0^x \frac{f_Y(y)}{(1-y)^\lambda}\,\mathrm{d}y - \lambda \frac{(1-x)^{\lambda-1}}{x} \frac{f_Y(x)}{(1-x)^\lambda} \ ,$$

which implies the following linear second-order differential equation for f_Y:

$$x(1-x)f_Y''(x) + [(\lambda-2)x+1]f_Y'(x) + \lambda f_Y(x) = 0 \ . \tag{30}$$

According to [36, Eq. 2.258], the left side of this equation is a total derivative, so the equation can be integrated once to the linear first-order differential equation

$$x(1-x)f_Y'(x) + \lambda x f_Y(x) = C_1 \ ,$$

where C_1 is an arbitrary constant. Since for $x \to 0$, the left-hand side of the last equation tends to zero, the constant C_1 is zero (see the parenthetical remark in the derivation of (25)). As a result, we arrive at the separable differential equation $\frac{\mathrm{d}f_Y}{\mathrm{d}x} = -\lambda \frac{f_Y(x)}{1-x}$, whose general solution is $f_Y(x) = C_2(1-x)^\lambda$. Normalizing, we obtain

$$f_Y(x) = (\lambda+1)(1-x)^\lambda \, \chi_{(0,1)}(x) \ . \tag{31}$$

The mean and the variance of Y are easy to find: $\mathbb{E}Y = \frac{1}{\lambda+2}$, $\operatorname{Var} Y = \frac{\lambda+1}{(\lambda+2)^2(\lambda+3)}$,

To compute f_X, we use (14) and integrate by parts:

$$\begin{aligned}
f_X(x) &= \frac{\chi_{(0,1)}(x)}{1-x} \int_0^x \mathrm{d}y\,(\lambda+1)(1-y)^\lambda \int_{\ln\frac{1-y}{1-x}}^\infty \mathrm{d}t\,\frac{\lambda \mathrm{e}^{-\lambda t}}{t} \\
&= \frac{\lambda\,\chi_{(0,1)}(x)}{1-x} \int_{y=0}^x \mathrm{d}\left[(1-y)^{\lambda+1}\right] \int_\infty^{\ln\frac{1-y}{1-x}} \mathrm{d}t\,\frac{\mathrm{e}^{-\lambda t}}{t} \\
&= \frac{\lambda\,\chi_{(0,1)}(x)}{1-x} \left[(1-y)^{\lambda+1} \int_\infty^{\ln\frac{1-y}{1-x}} \mathrm{d}t\,\frac{\mathrm{e}^{-\lambda t}}{t} \Bigg|_{y=0}^{x} + \int_0^x (1-y)^\lambda \left. \frac{\mathrm{e}^{-\lambda t}}{t} \right|_{t=\infty}^{\ln\frac{1-y}{1-x}} \mathrm{d}y \right] \\
&= \frac{\lambda\,\chi_{(0,1)}(x)}{1-x} \left\{ (1-x)^{\lambda+1} \left(\mathrm{li}\big(\tfrac{1}{1-x}\big) + \lim_{y\uparrow x} \left[\mathrm{li}\big(\big(\tfrac{1-x}{1-y}\big)^\lambda\big) - \mathrm{li}\big(\tfrac{1-y}{1-x}\big) \right] \right) - \mathrm{li}((1-x)^\lambda) \right\} \ ,
\end{aligned}$$

which simplifies to

$$f_X(x) = \frac{\lambda\,\chi_{(0,1)}(x)}{1-x} \left\{ (1-x)^{\lambda+1} \left[\mathrm{li}\big(\tfrac{1}{1-x}\big) + \ln\lambda \right] - \mathrm{li}\big((1-x)^\lambda\big) \right\} \ . \tag{32}$$

Here li is the logarithmic integral [35, Eqn. 5.1.3]. We plot f_X for several values of λ in Figure 2.

Long calculations yield the following expressions for the mean and the variance of X:

$$\mathbb{E}\,X = 1 - \frac{\lambda(\lambda+1)}{\lambda+2} \ln\frac{\lambda+1}{\lambda} \ , \tag{33}$$

$$\operatorname{Var} X = \lambda(\lambda+1) \left[\frac{1}{2(\lambda+3)} \ln\frac{\lambda+2}{\lambda} - \frac{\lambda(\lambda+1)}{(\lambda+2)^2} \left(\ln\frac{\lambda+1}{\lambda} \right)^2 \right] \ . \tag{34}$$

For very small positive λ, $\mathbb{E}[X] \approx 1 + \frac{1}{2}\lambda\ln\lambda$, $\operatorname{Var} X \approx \frac{1}{6}\lambda\ln\left(\frac{2}{\lambda}\right) - \frac{1}{4}(\lambda\ln\lambda)^2$; for very large λ, $\mathbb{E}[X] \approx \frac{3}{2\lambda} - \frac{17}{6\lambda^2}$, $\operatorname{Var} X \approx \frac{17}{12\lambda^2} - \frac{9}{\lambda^3}$. We show the plots of $\mathbb{E}[X] + \sigma_X$, $\mathbb{E}[X]$, and $\mathbb{E}[X] - \sigma_X$, where $\sigma_X = \sqrt{\operatorname{Var} X}$, as functions of λ, in Figure 3.

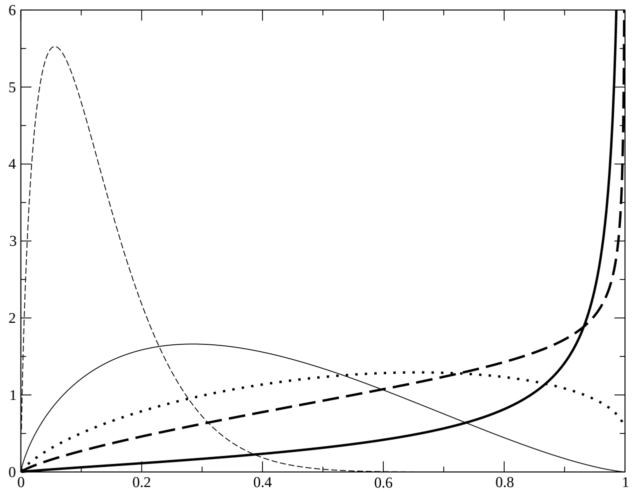

Figure 2: Plots of f_X for the case of growth with saturation, for $\lambda = 0.1$ (thick solid line), $\lambda = 0.5$ (thick dashed line), $\lambda = 0.9$ (dotted line), $\lambda = 2$ (thin solid line), $\lambda = 10$ (thin dashed line).

4.3. Case (C) – logistic equation

For the logistic equation (7),

$$\phi_t(x_0) = \frac{x_0}{x_0 + (1-x_0)\mathrm{e}^{-t}} \ , \qquad \psi(x_0, x) = \ln\left(\frac{1-x_0}{x_0}\,\frac{x}{1-x}\right) \ . \tag{35}$$

A straightforward computation yelds

$$\begin{aligned} f_{Y_{n+1}|Y_n}(x_{n+1}|x_n) &= \int_{\max\{x_n, x_{n+1}\}}^{1} \lambda \left(\frac{x_n}{1-x_n}\,\frac{1-x_{n+1}^-}{x_{n+1}^-}\right)^{\lambda} \frac{1}{(x_{n+1}^-)^2(1-x_{n+1}^-)}\,\mathrm{d}x_{n+1}^- \\ &= \begin{cases} \dfrac{\lambda + x_n}{(\lambda+1)x_n} & \text{if } 0 < x_{n+1} < x_n \ , \\ \left(\dfrac{x_n}{1-x_n}\,\dfrac{1-x_{n+1}}{x_{n+1}}\right)^{\lambda} \dfrac{\lambda + x_{n+1}}{(\lambda+1)x_{n+1}} & \text{if } 0 < x_n < x_{n+1} \ , \end{cases} \end{aligned}$$

so the stationary p.d.f. of the post-disturbance level Y satisfies the integral equation

$$f_Y(x) = \frac{\lambda + x}{(\lambda+1)x}\left(\frac{1-x}{x}\right)^{\lambda} \int_0^x \left(\frac{y}{1-y}\right)^{\lambda} f_Y(y)\,\mathrm{d}y + \frac{1}{\lambda+1}\int_x^1 \left(1 + \frac{\lambda}{y}\right) f_Y(y)\,\mathrm{d}y \ , \qquad x \in [0,1) \ . \tag{36}$$

The derivatives of f_Y can be found from (36) to be

$$f_Y'(x) = -\frac{\lambda\,(1-x)^{\lambda-1}}{x^{\lambda+2}} \int_0^x \left(\frac{y}{1-y}\right)^{\lambda} f_Y(y)\,\mathrm{d}y \ ,$$

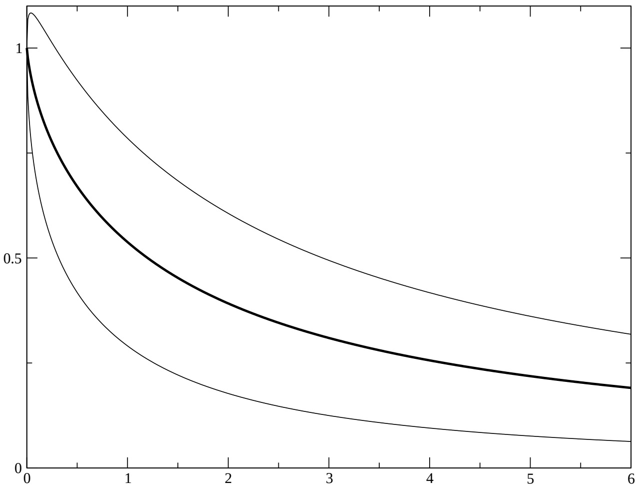

Figure 3: Plots of $\mathbb{E}[X]$ (thick solid line), $\mathbb{E}[X]+\sigma_X$, and $\mathbb{E}[X]-\sigma_X$, as functions of λ, for the case of growth with saturation.

$$f_Y''(x) = -\frac{\lambda}{x^2(1-x)}\, f_Y(x) - \frac{\lambda\,(1-x)^{\lambda-2}(3x-2-\lambda)}{x^{\lambda+3}} \int_0^x \left(\frac{y}{1-y}\right)^{\lambda} f_Y(y)\,\mathrm{d}y \ ,$$

which implies the following linear second-order differential equation for f_Y:

$$x^2(1-x)\, f_Y''(x) + (2+\lambda-3x)x\, f_Y'(x) + \lambda\, f_Y(x) = 0 \ , \qquad x \in [0,1) \ .$$

Following a suggestion in [36, Eq. 2.325a], we set $f_Y(x) = \frac{1}{x}u(x)$, and rewrite the differential equation for f_Y as a differential equation for u:

$$x(1-x)\, u''(x) + (\lambda - x)\, u'(x) + u(x) = 0 \ .$$

The left-hand side of this differential equation is a total derivative, so we integrate both sides to obtain

$$x(1-x)\, u'(x) + (x+\lambda-1)\, u(x) = C_1 \ ,$$

where C_1 is a constant. Setting $x=1$ in the left-hand side, and noticing that the integral equation (36) implies that $f_Y(1)=0$, we see that $C_1=0$, and obtain that u satisfies a separable differential equation with general solution $u(x) = C_2\,\frac{(1-x)^{\lambda}}{x^{\lambda-1}}$, hence $f_Y(x) = C_2\left(\frac{1-x}{x}\right)^{\lambda}$. Finally, f_Y must satisfy the normalization condition $\int_0^1 f_Y(x)\,\mathrm{d}x = C_2\int_0^1 \left(\frac{1-x}{x}\right)^{\lambda}\,\mathrm{d}x = 1$, but this integral converges only for $|\lambda| < 1$. Therefore, we obtain that

$$f_Y(x) = \frac{\sin(\pi\lambda)}{\pi\lambda}\left(\frac{1-x}{x}\right)^{\lambda} \qquad \text{if} \quad \lambda < 1 \ . \tag{37}$$

Note the interesting fact that for very small positive values of λ, the stationary distribution of Y tends to uniform. The mean and the variance of Y for $\lambda < 1$ are $\mathbb{E}\,Y = \frac{1-\lambda}{2}$ and $\mathrm{Var}\,Y = \frac{1-\lambda^2}{12}$, respectively.

The non-existence of a p.d.f. of a stationary distribution of the post-disturbance level for $\lambda \geq 1$ can be easily understood. A large value λ means that the disturbances occur frequently. Since for small values of x the function $g(x) = x(1-x)$ has small values, if X becomes small, then it cannot recover before the next disturbance occurs. Therefore, for $\lambda \geq 1$, the disturbances will eventually drive X and, hence, Y, to 0. In this case, there is a stationary distribution, but it is not described by a p.d.f.:

$$\mathbb{P}(Y = 0) = 1 \quad \text{if} \quad \lambda \geq 1 \; .$$

Remark 3. *For $\lambda < 1$, there exist two invariant measures of the post-disturbance levels – a discrete one (concentrated at 0), and an absolutely continuous one (given by the p.d.f.* (37)*). From the way we solve the integral equation* (11) *by reducing it to a differential equation, it is clear that under the conditions of Theorem 1, there is no more than one p.d.f. that solves* (11)*. This, however, does not exclude the possibility of existence of two invariant measures one of which is discrete, as in the present example.*

In the case $\lambda < 1$, we obtain from (15)

$$f_X(x) = \frac{\chi_{(0,1)}(x)}{x(1-x)} \int_0^\infty \mathrm{d}t \, \frac{\lambda \mathrm{e}^{-\lambda t}}{t} \int_{\phi_{-t}(x)}^x \mathrm{d}y \, \frac{\sin(\pi\lambda)}{\pi\lambda} \left(\frac{1-y}{y} \right)^\lambda \; . \tag{38}$$

We were not able to solve this integral analytically, but one can use it for numerical computations. We plot f_X for several values of λ in Figure 4.

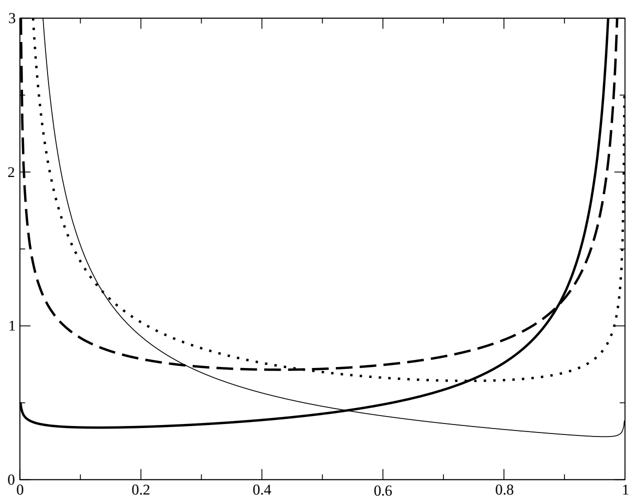

Figure 4: Plots of f_X for the logistic case, for $\lambda = 0.1$ (thick solid line), $\lambda = 0.3$ (thick dashed line), $\lambda = 0.5$ (dotted line), $\lambda = 0.7$ (thin solid line), $\lambda = 0.9$ (thin dashed line).

One can also use (38) in order to compute the moments of X numerically. For example, one can change the order of integration to find

$$
\begin{aligned}
\mathbb{E}\,X &= \int_0^1 x\, f_X(x)\, \mathrm{d}x \\
&= \frac{\sin(\pi\lambda)}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-\lambda t}}{t} \int_0^1 \frac{\mathrm{d}x}{1-x} \int_{\phi_{-t}(x)}^x \mathrm{d}y\, \left(\frac{1-y}{y}\right)^\lambda \\
&= \frac{\sin(\pi\lambda)}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-\lambda t}}{t} \int_0^1 \mathrm{d}y\, \left(\frac{1-y}{y}\right)^\lambda \int_y^{\phi_t(y)} \frac{\mathrm{d}x}{1-x} \\
&= \frac{\sin(\pi\lambda)}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-\lambda t}}{t} \int_0^1 \mathrm{d}y\, \left(\frac{1-y}{y}\right)^\lambda \ln(1+y\mathrm{e}^t - y) \\
&= \frac{\lambda(1-\lambda)}{2} \int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-\lambda t}(\mathrm{e}^t-1)}{t}\, {}_3F_2(\{1,1,2-\lambda\};\{2,3\};1-\mathrm{e}^t)\ ,
\end{aligned}
$$

where ${}_pF_q$ is the generalized hypergeometric function, [35, Eqn. 15.1.1].

$$
{}_pF_q(\mathbf{a};\mathbf{b};z) = \sum_{k=0}^\infty \frac{(a_1)_k (a_2)_k \cdots (a_p)_k}{(b_1)_k (b_2)_k \cdots (b_q)_k}\, \frac{z^k}{k!}\ ,
$$

where $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$ is the *Pochhammer symbol*. In our particular case,

$$
{}_3F_2(\{1,1,2-\lambda\};\{2,3\};1-\mathrm{e}^t) = 2\sum_{k=0}^\infty \frac{(1-\mathrm{e}^t)^k}{(k+1)(k+2)}\ .
$$

With the same change of variables, the second moment of X can be computed numerically from

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 x^2\, f_X(x)\, \mathrm{d}x \\
&= \frac{\sin(\pi\lambda)}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-\lambda t}}{t} \int_0^1 \frac{x\,\mathrm{d}x}{1-x} \int_{\phi_{-t}(x)}^x \mathrm{d}y\, \left(\frac{1-y}{y}\right)^\lambda \\
&= \frac{\sin(\pi\lambda)}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-\lambda t}}{t} \int_0^1 \mathrm{d}y\, \left(\frac{1-y}{y}\right)^\lambda \int_y^{\phi_t(y)} \frac{x\,\mathrm{d}x}{1-x} \\
&= \frac{\sin(\pi\lambda)}{\pi} \int_0^\infty \mathrm{d}t\, \frac{\mathrm{e}^{-\lambda t}}{t} \int_0^1 \mathrm{d}y\, \left(\frac{1-y}{y}\right)^\lambda \left[\ln(1+y\mathrm{e}^y - y) + \frac{y(y-1)(1-\mathrm{e}^{-t})}{y+(1-y)\mathrm{e}^{-t}}\right]\ .
\end{aligned}
$$

We computed $E[X^2]$ numerically by using the last representation (as a double integral). We show the plots of $\mathbb{E}[X]+\sigma_X$, $\mathbb{E}[X]$, and $\mathbb{E}[X]-\sigma_X$ (with $\sigma_X = \sqrt{\mathrm{Var}\, X}$), as functions of λ, in Figure 5.

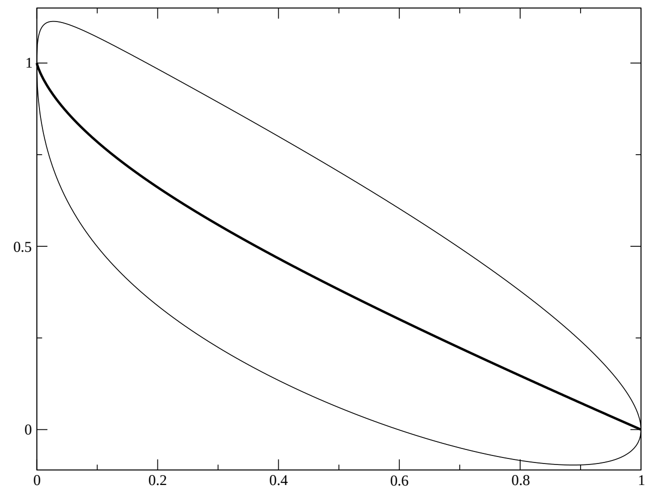

Figure 5: Plots of $\mathbb{E}[X]$ (thick solid line), $\mathbb{E}[X]+\sigma_X$, and $\mathbb{E}[X]-\sigma_X$, as functions of λ, for the logistic case.

4.4. Case (D) – exponential growth: an example without an absolutely continuous invariant measure

The exponential growth equation (8) yields

$$\phi_t(x_0) = x_0\,\mathrm{e}^t\ , \qquad \psi(x_0,x) = \ln\frac{x}{x_0}\ . \tag{39}$$

The conditional p.d.f. connecting the p.d.f.s of Y_n and Y_{n+1} has the form

$$f_{Y_{n+1}|Y_n}(x_{n+1}|x_n) = \begin{cases} \dfrac{\lambda}{(\lambda+1)x_n} & \text{if } 0 < x_{n+1} < x_n\ , \\ \dfrac{\lambda x_n^\lambda}{(\lambda+1)x_{n+1}^{\lambda+1}} & \text{if } 0 < x_n < x_{n+1}\ , \end{cases}$$

and the stationary p.d.f. f_Y satisfies the integral equation

$$f_Y(x) = \frac{\lambda}{\lambda+1}\left[\frac{1}{x^{\lambda+1}}\int_0^x y^\lambda f_Y(y)\,\mathrm{d}y + \int_x^\infty \frac{f_Y(y)}{y}\,\mathrm{d}y\right]\ .$$

This equation can be converted to the Cauchy-Euler equation

$$x^2 f_Y''(x) + (\lambda+2)x f_Y'(x) + \lambda f_Y(x) = 0\ ,$$

whose general solution is well-known:

$$f_Y(x) = \begin{cases} C_1 x^{-1} + C_2 x^{1-\lambda} & \text{if } \lambda \neq 1\ , \\ C_1 x^{-1} + C_2 x^{-1}\ln x & \text{if } \lambda = 1\ . \end{cases}$$

One can easily check that this function cannot satisfy the normalization condition for any λ, so an absolutely continuous invariant measure in this case does not exist. The non-existence of a continuous stationary p.d.f. f_Y can be easily explained intuitively – if at some moment the value of X is too small, then the growth rate is small and the solution is driven to 0 by the disturbances; if the solution becomes too large, then the disturbance cannot compensate the fast growth and X drifts away to infinity.

5. Comparison of the "standard" continuous-time discrete-state space death-immigration-disaster process with its continuous analogue

In this section we compare the classical example of a continuous-time discrete-state space model of a death-immigration-disaster process with its continuous counterpart, as an illustration of the differences in the predictions of the two models.

Consider the "standard" continuous-time discrete-state space death-immigration-disaster process with state space $\mathcal{S} = \{0, 1, 2, 3, \ldots\}$. Let the immigration rate be ν, the death rate from state n be $n\mu$, and the disaster rate be δ, then the generator of the evolution semigroup of the system is given by

$$\mathbf{G} = \begin{pmatrix} -\nu & \nu & 0 & 0 & 0 & \cdots \\ \delta+\mu & -(\delta+\mu+\nu) & \nu & 0 & 0 & \cdots \\ \delta & 2\mu & -(\delta+2\mu+\nu) & \nu & 0 & \cdots \\ \delta & 0 & 3\mu & -(\delta+3\mu+\nu) & \nu & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} . \tag{40}$$

Our goal in this section is to formulate a continuous-state space continuous-time stochastic process that is deterministic between the disasters that occur at random times, as in our setup described in the Introduction, and to compare some quantities characterizing these two processes. First we need to write a differential equation after an appropriate rescaling. Let $X_{\rm ph}(t_{\rm ph})$ be the discrete state-space process; here the subscript "ph" stands for "physical". We choose a large positive dimensionless constant N and define $\widetilde{X} := \frac{1}{N}X_{\rm ph}$, and we will think of $\widetilde{X}$ as taking a continuum of values. Due solely to immigration, $\widetilde{X}$ will increase on average by $\frac{\nu}{N}\Delta t_{\rm ph} + o(\Delta t_{\rm ph})$ in a short time interval of duration $\Delta t_{\rm ph}$. Due solely to death, $\widetilde{X}$ will decrease on average by $\frac{\mu X_{\rm ph}}{N}\Delta t_{\rm ph} + o(\Delta t_{\rm ph}) = \mu\widetilde{X}\,\Delta t_{\rm ph} + o(\Delta t_{\rm ph})$ in a short time interval of duration $\Delta t_{\rm ph}$. Therefore, we can write

$$\frac{\Delta \widetilde{X}}{\Delta t_{\rm ph}} \approx \frac{\nu}{N} - \mu\widetilde{X} \ . \tag{41}$$

Finally, we eliminate the rate μ by defining a dimensionless time by $t := \mu\, t_{\rm ph}$, which, in the limit $\Delta t \to 0$, transforms (41) into the differential equation

$$\frac{\rm d}{{\rm d}t}\widetilde{X} = \alpha - \widetilde{X} \ , \qquad \alpha := \frac{\nu}{N\mu} \ , \tag{42}$$

describing the evolution of the system between two consecutive disasters. The times of occurrence of the disasters form a Poisson process with (dimensionless) rate $\lambda = \delta/\mu$.

Chao and Zheng [37] (see also Kyriakidis [38]) solved the general problem of determining the transient probabilities of a process with generator $\mathbf{G}$ given by (40) (different approaches to such problems are discussed in the review [39] of Economou and Fakinos). In particular, they obtained that the stationary distribution $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$ of $X_{\rm ph}$ is

$$\begin{aligned} \pi_n &= \frac{1}{n!}\left(\frac{\nu}{\mu}\right)^n \mathrm{e}^{-\nu/\mu} \int_0^1 \exp\left(\frac{\nu}{\mu} y^{\mu/\delta}\right) \left(1 - y^{\mu/\delta}\right)^n \,\mathrm{d}y \\ &= \left(\frac{\nu}{\mu}\right)^n \mathrm{e}^{-\nu/\mu} \frac{\Gamma\left(1 + \frac{\delta}{\mu}\right) {}_1F_1\left(\frac{\delta}{\mu}; n + 1 + \frac{\delta}{\mu}; \frac{\nu}{\mu}\right)}{\Gamma\left(n + 1 + \frac{\delta}{\mu}\right)} \;, \end{aligned} \tag{43}$$

where ${}_1F_1$ is the Kummer confluent hypergeometric function (see [35, Eqn. 13.1.2], where ${}_1F_1(a; b; z)$ is denoted by $M(a, b, z)$). Using (43), one can compute the mean and the variance of the stationary distribution $\boldsymbol{\pi}$ of $X_{\rm ph}$:

$$\mathbb{E}[X_{\rm ph}] = \frac{\nu}{\mu + \delta} \;, \qquad \operatorname{Var} X_{\rm ph} = \frac{\nu}{\delta} \frac{1 + 2\left(\frac{\mu}{\delta}\right)^2 + \frac{\mu}{\delta}\left(3 + \frac{\nu}{\delta}\right)}{\left(1 + \frac{\mu}{\delta}\right)^2 \left(1 + 2\frac{\mu}{\delta}\right)} \;.$$

In our notations these formulas read

$$\mathbb{E}[\widetilde{X}] = \frac{\mathbb{E}[X_{\rm ph}]}{N} = \frac{\alpha}{1 + \lambda} \;, \qquad \operatorname{Var} \widetilde{X} = \frac{\operatorname{Var} X_{\rm ph}}{N^2} = \frac{\alpha^2 \lambda}{(\lambda + 1)^2 (\lambda + 2)} + \frac{\alpha}{N(\lambda + 1)} \;. \tag{44}$$

Now we treat the process as a continuous-state space process X that satisfies the differential equation $\frac{\mathrm{d}}{\mathrm{d}t} X = \alpha - X$ (cf. (42)), and the disturbances occur as a Poisson process of rate $\lambda = \delta/\mu$, and are complete (i.e., $\mathbb{P}(U = 0) = 1$, or, formally, f_U is the Dirac delta function, $f_U(x) = \delta(x)$). Then $\phi_t(x_0) = \alpha + (x_0 - \alpha)\mathrm{e}^{-t}$, $\psi(x_0, x) = \ln \frac{x_0 - \alpha}{x - \alpha}$, and the expression (15) for f_X yields

$$\begin{aligned} f_X(x) &= \frac{\chi_{(0,\alpha)}(x)}{\alpha - x} \int_0^\infty \mathrm{d}t \, \frac{\lambda \mathrm{e}^{-\lambda t}}{t} \int_{\max\{0, \alpha + (x - \alpha)\mathrm{e}^t\}}^x \mathrm{d}y \, \delta(y) \\ &= \frac{\chi_{(0,\alpha)}(x)}{1 - x} \int_0^\infty \mathrm{d}t \, \frac{\lambda \mathrm{e}^{-\lambda t}}{t} \chi_{(-\ln \frac{\alpha - x}{\alpha}, \infty)}(t) \;, \\ &= -\frac{\lambda}{\alpha - x} \operatorname{li}\left((1 - \tfrac{x}{\alpha})^\lambda\right) \chi_{(0,\alpha)}(x) \;. \end{aligned} \tag{45}$$

The mean and the variance of X are

$$\mathbb{E}\, X = \alpha \left(1 - \lambda \ln \frac{\lambda + 1}{\lambda}\right) \;, \qquad \operatorname{Var} X = \alpha^2 \lambda \left[\frac{1}{2} \ln \frac{\lambda + 2}{\lambda} - \lambda \left(\ln \frac{\lambda + 1}{\lambda}\right)^2\right] \;. \tag{46}$$

In Figure 6 we plot the probability mass functions (43) of $\widetilde{X}$ for $N = 10$, 50, and 3000, and the p.d.f. (45) of X, all for $\lambda = 0.1$, $\alpha = 1.0$. In Figure 7 we display the probability

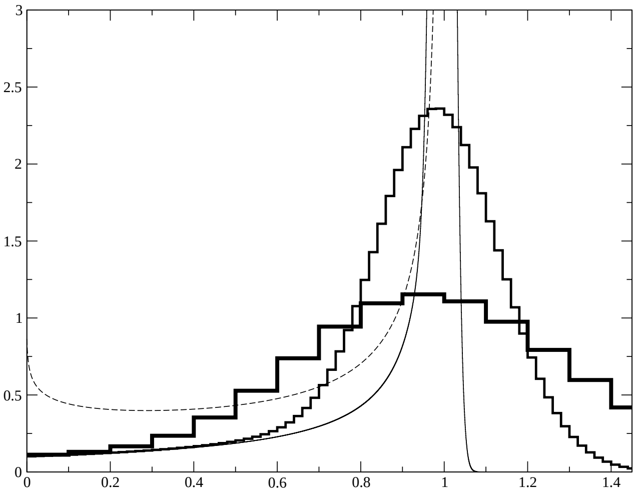

Figure 6: P.d.f. of X (dashed line) and p.m.f.'s of $\widetilde{X}$ for $N = 10$ (very thick stairs), 50 (thick stairs), and 3000 (thin stairs), all for $\lambda = 0.1$, $\alpha = 1.0$.

mass functions (43) of $\widetilde{X}$ for $N = 10$, 50, and 3000, and the p.d.f. (45) of X, all for $\lambda = 1.1$, $\alpha = 1.0$.

Figures 8 and 9 show the means and the variances of X (44) and $\widetilde{X}$ (46) as functions of λ, all for $\alpha = 1$. Note that the mean of $\widetilde{X}$ does not depend on N, while in Figure 9 we plotted $\operatorname{Var}\widetilde{X}$ for $N = 10$, 50, and 1000.

One important difference between the discrete- and the continuous-state space cases is that, in the limit of absence of disasters, $\lim_{\lambda\to 0} \operatorname{Var} X = 0$, while $\lim_{\lambda\to 0} \operatorname{Var}\widetilde{X} = \frac{\alpha}{N} = \frac{\nu}{\mu}$. This is easy to explain – while $X(t)$ tends to α as $t \to \infty$, $\widetilde{X}(t)$ fluctuates because of the immigration and death processes.

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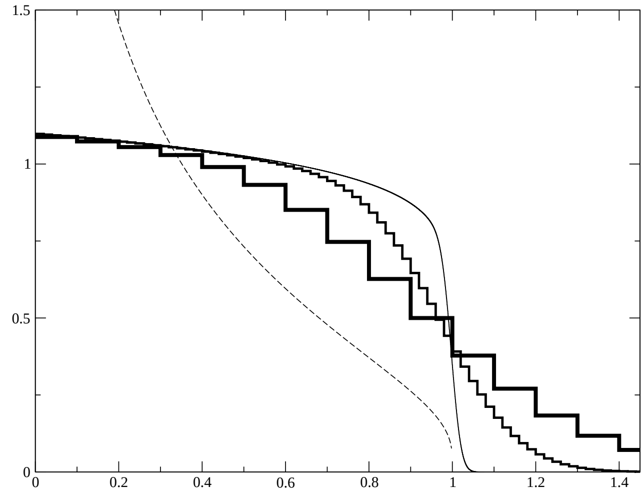

Figure 7: P.d.f. of X (dashed line) and p.m.f.'s of $\widetilde{X}$ for $N = 10$ (very thick stairs), 50 (thick stairs), and 3000 (thin stairs), all for $\lambda = 1.1$, $\alpha = 1.0$.

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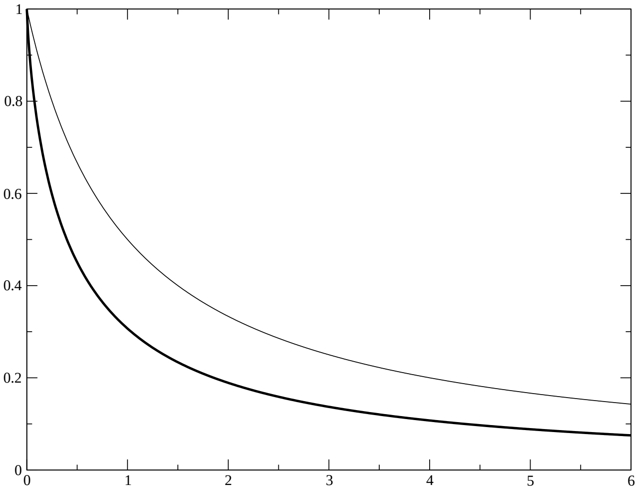

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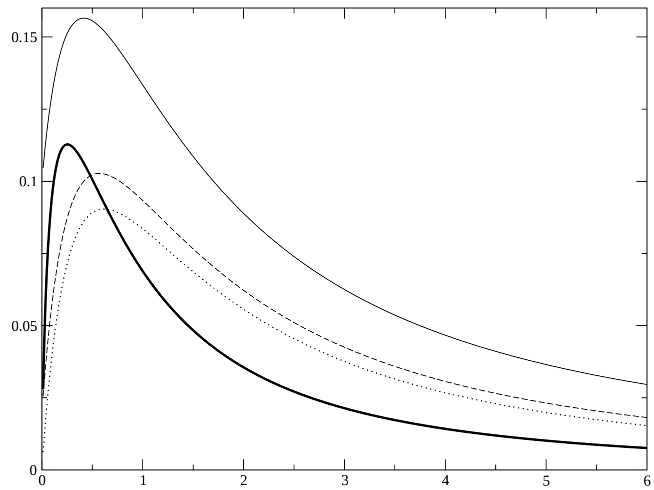

Figure 9: Variances of X (thick line) and $\widetilde{X}$ (for $N = 10$–thin solid line, $N = 50$–dashed line, and $N = 3000$–dotted line), as functions of λ, all for $\alpha = 1$.

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