M(λ)/M(μ) 1 queue

Problem 2. In this problem you will find the probability of ruin of a gambler for the same transition probability matrix as in Problem 1, but using different methods.

Recall Theorem 3.2.2 of the book according to which, if C is a closed and irreducible set of recurrent states, and D is the set of all transient states, and $i \in D$ is a transient state, then the probability $r_i(C)$ of the chain to enter the set C eventually if $X_0 = i$ is the smallest nonnegative solution of the system

$$r_i(C) = \sum_{j \in C} p_{ij} + \sum_{j \in D} p_{ij}\, r_j(C) \ , \qquad \text{for all } \ i \in D \ .$$

Moreover, if D is a finite set (as in this problem), then the solution is unique.

(a) Write down this system of equations for the probabilities $r_i(\{0\})$ of eventual ruin starting from state i.

(b) Solve the system derived in (a).

(c) Compare your results with your results from Problem 1. Please be specific what numbers you are comparing.

Problem 3. Imagine a barber's shop with one barber and with a very large waiting room (so that every coming customer stays in the waiting room waiting to be served).

Assume that the customers arrive at the barber's shop in the manner of a Poisson process with rate λ. The time it takes the barber to serve each customer is an exponentially distributed random variable with parameter μ; we can assume that the times it takes the barber to shave different customers are independent of each other and of the arrival process.

Let X_t be the number of customers in the shop at time t (either waiting or being shaved by the barber). Assume that $X_0 = 0$ (i.e., that there are no customers when the barber opens the shop).

One can easily see that $X = \{X_t : t \geq 0\}$ is a continuous-time Markov process on the state space $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$ with generator

$$\mathbf{G} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \cdots \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \cdots \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

For continuous-time Markov processes one can also define stationary distributions $\boldsymbol{\pi} = (\pi_0, \pi_1, \pi_2, \ldots)$, such that $\boldsymbol{\pi} = \boldsymbol{\pi}\mathbf{P}_t$ for each $t \geq 0$. This condition is equivalent to the condition $\boldsymbol{\pi}\mathbf{G} = \mathbf{0}$, where $\mathbf{G}$ is the generator of the Markov process, and $\mathbf{0}$ is the zero vector-row; please use this condition to find the stationary distribution below (soon we will prove it in class).

If the Markov process is irreducible (as in this problem) and there exists a stationary distribution, then a theorem (analogous to the Ergodic Theorem in the discrete-time case) guarantees that this distribution is unique, and that

$$\lim_{t \to \infty} p_{ij}(t) = \pi_j \ .$$

This means that under these conditions, in the long run, π_i is the fraction of time the Markov process spends in state j (i.e., the probability to find the system in state j if we observe it at a random moment is π_j).

(a) Show by using induction that the components of the stationary distribution π of the Markov process X satisfy

$$\frac{\pi_{j+1}}{\pi_j} = \frac{\lambda}{\mu} .$$

(b) Prove that the stationary distribution π is given by $\pi_j = \left(1 - \frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j$.

(c) What condition should λ and μ satisfy for the stationary distribution π to exist? Having found this condition, please try to explain it intuitively in a couple of sentences.

For the rest of the problem, assume that λ and μ are such that π exists, and that Mr. Masterson arrived at the barber's shop very long time after the barber opened the shop.

(d) What is the probability that at the time of arrival of Mr. Masterson in the barber's shop there are exactly j customers in the shop (for $j \in \mathbb{Z}_+$)?

Hint: This question is very easy; think about the mathematical meaning of the words "very long time after the barber opened the shop".

(e) Let W be the *waiting time* of Mr. Masterson, i.e., the time he has to wait until the barber starts shaving him (but *not* including the time it will take the barber to shave him!). The random variable W is neither discrete, nor continuous, because there is a non-zero probability that $W = 0$, but if we know that $W > 0$, then W has continuous distribution. What is the probability that $W = 0$?

(f) Mr. Masterson enters the barber's shop and sees that there is one customer being shaved and $j - 1$ customers waiting (i.e., that the total number of customers in the store is $j \geq 1$). Explain why the time that Mr. Masterson should expect to wait before the barber starts shaving him (i.e., the expected value of the waiting time W) is equal to $\frac{j}{\mu}$.

(g) Before entering the barber's shop (i.e., before he sees how many customers are in the store), Mr. Masterson is trying to estimate how long he will be waiting in the store before the barber starts shaving him. Can you help him with the calculation? (Assume that Mr. Masterson is a regular customer, so that he knows the values of the constants λ and μ.)

Hint: Use the Tower Rule in the form

$$E[W] = E\left[E[W|X_t]\right] = \sum_{j=0}^{\infty} E[W|X_t = j]\, P(X_t = j) ,$$

for very large t; you have already found $E[W|X_t = j]$ and $P(X_t = j)$ in the limit $t \to \infty$.

Problem 3

(a) • Starting step: from $\underline{\pi}\,\underline{\underline{G}} = \underline{0}$, we obtain

$$\pi_0 \nu_{00} + \pi_1 \nu_{10} + \pi_2 \nu_{20} + \cdots = 0$$

$$\Longrightarrow \pi_0(-\lambda) + \pi_1 \mu = 0$$

$$\Longrightarrow \frac{\pi_1}{\pi_0} = \frac{\lambda}{\mu}$$

• Inductive step: assume that for some $k \geq 1$,

$$\frac{\pi_{k+1}}{\pi_k} = \frac{\lambda}{\mu}.$$

From $\underline{\pi}\,\underline{\underline{G}} = \underline{0}$, we have

$$\lambda \pi_k - (\mu+\lambda)\pi_{k+1} + \mu \pi_{k+2} = 0;$$

divide by π_{k+1}:

$$\mu \frac{\pi_{k+2}}{\pi_{k+1}} = -\lambda \frac{\pi_k}{\pi_{k+1}} + \mu + \lambda$$

$$= \cancel{\lambda}\frac{\mu}{\cancel{\lambda}} + \cancel{\mu} + \lambda$$

inductive assumption

$$= \lambda$$

$$\Longrightarrow \frac{\pi_{k+2}}{\pi_{k+1}} = \frac{\lambda}{\mu}.$$

(b) $\pi_1 = \frac{\lambda}{\mu}\pi_0$

$$\pi_2 = \frac{\lambda}{\mu}\pi_1 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0$$

$$\vdots$$

$$\pi_j = \left(\frac{\lambda}{\mu}\right)^j \pi_0$$

$$\vdots$$

$$1 = \sum_{j=0}^{\infty} \pi_j = \pi_0 \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j = \frac{\pi_0}{1-\frac{\lambda}{\mu}}$$

$$\Longrightarrow \pi_0 = \left(1 - \frac{\lambda}{\mu}\right),$$

$$\pi_j = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j, \quad j \in \mathbb{Z}_+.$$

(c) From (b) we see that for $\underline{\pi}$ to exist, we must have $1 - \frac{\lambda}{\mu} > 0$, i.e.,

$$\lambda < \mu.$$

The meaning of this condition is quite clear — a stationary situation can be achieved only if the rate λ of the customers coming in the shop is smaller than the rate at which the barber shaves them, otherwise the number of waiting customers will grow with time (on average) and "equilibrium" will never be achieved.

(d) For very large t (when the stationary distribution is achieved),

$$P(X_t = j) \approx \pi_j = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j, \quad j \in \mathbb{Z}_+.$$

(e) $P(W=0) = P(X_t = 0)$ ← for very large t

$$\approx \pi_0 = 1 - \frac{\lambda}{\mu}$$

(f) let $E_1, E_2, \ldots, E_j$ are the times that the barber will need to shave each of the j customers in the store. Note that these random variables are of type $Exp(\mu)$, so that they are memoryless – we need the memorylessness property to assert that the time the barber will need to finish the customer who is already being shaved, E_1, has the same distribution as $E_2, \ldots, E_j$. We know that $E[E_k] = \frac{1}{\mu}$, therefore

$$E[W] = E[E_1 + \cdots + E_j] = \sum_{k=1}^{j} E[E_k] = j \cdot \frac{1}{\mu} = \frac{j}{\mu}.$$

(g) $E[W] = E[E[W|X_t]]$

$$= \sum_{j=0}^{\infty} \underbrace{E[W|X_t=j]}_{=\frac{j}{\mu}\ (\text{part (f)})} \underbrace{P(X_t=j)}_{=\pi_j = \left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j\ (\text{parts (b), (d)})}$$

$$= \sum_{j=0}^{\infty} \frac{j}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^j$$

$$= \frac{1}{\mu}\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) \sum_{j=\cancel{0}\,1}^{\infty} j \left(\frac{\lambda}{\mu}\right)^{j}$$

$$= \frac{\lambda}{\mu^2}\left(1-\frac{\lambda}{\mu}\right) \frac{d}{dq}\left(\sum_{j=0}^{\infty} q^j\right)\Bigg|_{q=\frac{\lambda}{\mu}}$$

$$= \frac{\lambda}{\mu^2}\left(1-\frac{\lambda}{\mu}\right) \frac{d}{dq}\left(\frac{1}{1-q}\right)\Bigg|_{q=\frac{\lambda}{\mu}}$$

$$= \frac{\lambda}{\mu^2}\cancel{\left(1-\frac{\lambda}{\mu}\right)} \frac{1}{\left(1-\frac{\lambda}{\mu}\right)^{\cancel{2}}},$$

which, after elementary algebra, becomes

$$E[W] = \frac{\lambda}{\mu(\mu-\lambda)}.$$