Dimensional Reduction of Invariant Fields and Differential Operators. I. Reduction of Invariant Fields

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Mathematics Subject Classification (2010). Primary 53C80; Secondary 58Z05, 37J15, 58D19 70S10.

Keywords. Group action and symmetry; Invariant fields; Non-compact dimensional reduction.

Abstract. Problems related to symmetries and dimensional reduction are common in the mathematical and physical literature, and are intensively studied presently. As a rule, the symmetry group ("reducing group") and its orbits ("external dimensions") are compact, and this is essential in models where the volume of the orbits is related to physical quantities. But this case is only a part of the natural problems related to dimensional reduction.

In the present paper, we consider an action of a (generally non-compact) Lie group on a vector bundle, construct a formalism of reduced bundles for description of all invariant sections of the original bundle, and study the algebraic structures that occur in the reduced bundle. We show that in the case of a non-compact reducing group it is possible that the reduction is non-standard ("non-canonical"), and construct an explicit obstruction for canonical reduction in terms of cohomology of groups. We consider in detail the reduction of tangent and cotangent bundles, and show that, in general, the duality between the two is violated in the process of reduction. The reduction of the tensor product of tangent and cotangent bundles is also discussed. We construct examples of non-canonical dimensional reduction and of violation of duality between the tangent and cotangent bundles in the reduction.

1. Introduction

In this paper we present a geometric methodology for dimensional reduction of objects invariant under an action of a Lie group. Techniques for dimensional reduction exist in the literature, but in many of them the geometric aspects are not quite explicit or they treat particular problems, while our approach is quite general. Clearly, the price we pay for the generality of our approach is that sometimes the information we obtain is not as detailed as from the particular methods designed for attacking one particular problem.

Here is our general setup. We consider a Lie group G acting by bundle morphisms on a vector bundle ξ over a finite-dimensional base B (all objects and mappings are assumed to be C^∞). Let $C^\infty(\xi)^G$ stand for the set of all sections of ξ that are invariant with respect to the action of G. We propose a natural geometric procedure for constructing a *reduced bundle* ξ^G, the set of whose sections, $C^\infty(\xi^G)$, is in a bijective correspondence with the G-invariant sections of ξ, $C^\infty(\xi)^G$.

If some general assumptions on the group actions (Conditions A and B in Section 2.2) are satisfied, the reduced bundle can be constructed explicitly. The reduced bundle is constructed from local charts, so that we do not require that ξ or ξ^G be globally trivial. We prove that the arbitrariness in the choice of local charts in the construction of ξ^G does not affect the global object.

In the process of dimensional reduction of short exact sequences of vector bundles we need to use some facts from theory of cohomology of groups (briefly reviewed in the paper). It turns out that sometimes the reduced bundle has a simple structure – or, as we say, in this case the dimensional reduction is *canonical* (the rigorous definition is given in Section 3.3). We give explicitly the obstruction to canonical reduction in terms of certain cohomology groups, and show that for a compact group G this obstruction is zero, hence for compact G the dimensional reduction is always canonical. We construct a simple explicit example of a non-canonical reduction.

We pay special attention to the case of dimensional reduction of the tangent and cotangent bundles and their tensor products. In the case of a non-compact group G, the reductions of the tangent and cotangent bundles may exhibit some interesting phenomena, which is illustrated on a particular example.

In a second part of the paper (in preparation), we develop a technique for dimensional reduction of invariant differential operators based on the jet bundle description. Within the jet bundle language, the geometry of dimensional reduction becomes very explicit, and the operations on differential operators are reduced to simple algebraic manipulations.

The main novelty in our paper is establishing a clear connection between the procedure of dimensional reduction and cohomology of groups (Theorem 3.7) that makes explicit the distinction between the cases of canonical and non-canonical dimensional reduction (Definition 3.6 specifies the meaning of "canonical"). In particular, we construct explicitly an obstruction to canonical dimensional reduction and an example of a non-canonical reduction. Another novelty is the complete treatment of the dimensional reduction of the tangent and cotangent bundles to a smooth manifold and their tensor products (undertaken in Section 4).

Some of the techniques developed in this paper and its sequel have been used in our papers [1, 2, 3, 4, 5], but in this paper we consider the problem in full generality. We have not, for example, considered particular cases when the vector bundles are of a certain type or are endowed with some particular structure. In the procedure described in this paper, the arbitrariness in the choice of a base of the reduced manifold is not essential for the procedure – choosing a different base would amount merely to a reparametrization, but this would not change the essential features of the reduced objects. However, this choice would be important if additional structures are present in the original setup – for example, in the case of Riemannian manifolds, there will be other things that have to be taken into account, like the equations of Gauss-Codazzi-Mainardi related to the isometric embeddings of Riemannian manifolds (see [6, Sections 3–5]; these issues are the subject of the recent paper [7]). We do not discuss such issues in the present paper.

We have not attempted to survey the vast literature related to reduction for two reasons – firstly, the amount of literature makes it impossible for us to give proper credit, secondly, we are still trying to understand the relation between our method and the methods of other authors, and hope to discuss these in future publications. We only point out how some classical examples of dimensional reduction are related to the techniques developed in this paper.

We hope that our methods shed new light on the problem of symmetry reduction in many contexts. In particular, we believe that they would be useful in the study of the new structures occurring in the process of dimensional reduction (problems of Kaluza-Klein type), as well as for physical applications in the case of non-canonical reduction of invariant tensor fields.

The paper is organized as follows. Section 2 is devoted to a detailed explanation of the concept of a reduced vector bundle and the conditions we impose on the actions of the Lie group on the bundles in order for our construction to work. In Section 3 we introduce some facts from cohomology of groups and their use in the reduction procedure, and in Section 4 we apply this procedure to the case of reduction of the tangent and cotangent bundles and their tensor products.

Throughout the paper we assume that all manifolds, bundles, and maps are smooth (C^∞), even if this is not said explicitly.

2. Reduced vector bundles

2.1. Local chart description of vector bundles

In this section we give the basic definitions and set up the notations concerning the description of a vector bundle in coordinate charts, referring the reader to [8, Section I.3] for more details.

Let $\xi = (E, \pi, B)$ be a finite-dimensional vector bundle over the finite-dimensional manifold B. Let $\xi_b := \pi^{-1}(b)$ be the fiber of ξ over the point $b \in B$.

We will often need to restrict the base of a vector bundle to some submanifold. Let $C \subseteq B$ be a submanifold of the base B of ξ, and let $i : C \hookrightarrow B$ be the natural embedding. Then by ξ_C, or sometimes $\xi\restriction_C$ for clarity, we will denote the bundle $i^*\xi$ induced by i; in other words, $\xi_C = (E', \pi', C)$ where $E' := \pi^{-1}(C)$, and π' is the restriction of π to E'.

Let $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ be an open cover of the manifold B, and $\sigma_\alpha : U_\alpha \to \widetilde{U}_\alpha$ be diffeomorphisms from U_α to some manifolds $\widetilde{U}_\alpha$. Whenever $U_\alpha \cap U_\beta \neq \emptyset$, we set

$$\widetilde{U}_{\alpha,\beta} := \sigma_\alpha(U_\alpha \cap U_\beta) \subseteq \widetilde{U}_\alpha \tag{2.1}$$

and

$$f_{\alpha\beta} := \sigma_\alpha \circ \sigma_\beta^{-1} : \widetilde{U}_{\beta,\alpha} \to \widetilde{U}_{\alpha,\beta} \ . \tag{2.2}$$

In this case we say that the manifold B is obtained from the manifolds $\{\widetilde{U}_\alpha\}_{\alpha\in\mathscr{A}}$ through *gluing* (or *clutching* in the terminology of [8]) them by the diffeomorphisms $f_{\alpha\beta}$.

For each $\alpha \in \mathscr{A}$, let $\xi_\alpha := (E_\alpha, \pi_\alpha, \widetilde{U}_\alpha)$ be a vector bundle over $\widetilde{U}_\alpha$, and $\xi_{\alpha,\beta} := \xi_\alpha\restriction_{\widetilde{U}_{\alpha,\beta}}$. Let for each pair of indices α and β in $\mathscr{A}$ for which $U_\alpha \cap U_\beta \neq \emptyset$, there exists a vector bundle isomorphism

$$\phi_{\alpha\beta} := (F_{\alpha\beta}, f_{\alpha\beta}) : \xi_{\beta,\alpha} \to \xi_{\alpha,\beta} \tag{2.3}$$

over $f_{\alpha\beta}$ (where $F_{\alpha\beta} : \pi_\beta^{-1}(\widetilde{U}_{\beta,\alpha}) \to \pi_\alpha^{-1}(\widetilde{U}_{\alpha,\beta})$ and $\pi_\alpha \circ F_{\alpha\beta} = f_{\alpha\beta} \circ \pi_\beta$). Let the transition isomorphism $\phi_{\alpha\beta}$ satisfy the *cocycle conditions*

$$\phi_{\alpha\beta} \circ \phi_{\beta\gamma} = \phi_{\alpha\gamma} \tag{2.4}$$

whenever $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. We shall call the bundles ξ_α *coordinate bundles.* The transition isomorphisms $\phi_{\alpha\beta}$ glue these coordinate bundles into a vector bundle over B. More precisely, there exists a unique (up to isomorphism) vector bundle ξ over B and isomorphisms $\phi_\alpha : \xi_\alpha \to \xi\restriction_{U_\alpha}$ such that the diagram

$$\begin{tikzcd}
\xi_{\alpha,\beta} \arrow[dr, "\phi_\alpha"'] & & \xi_{\beta,\alpha} \arrow[ll, "\phi_{\alpha\beta}"'] \arrow[dl, "\phi_\beta"] \\
& \xi\restriction_{U_\alpha \cap U_\beta} &
\end{tikzcd} \tag{2.5}$$

commutes (see Theorem 3.2 in Chapter I of [8]). Clearly, ξ and the coordinate bundles ξ_α have the same standard fiber.

In these notations, defining a section $\psi \in C^\infty(\xi)$ is equivalent to defining sections $\psi_\alpha \in C^\infty(\xi_\alpha)$ for all $\alpha \in \mathscr{A}$ such that ψ_α are compatible with the transition isomorphisms:

$$\psi_\alpha = \phi_{\alpha\beta}(\psi_\beta) := F_{\alpha\beta} \circ \psi_\beta \circ f_{\alpha\beta}^{-1} \ . \tag{2.6}$$

Similarly, one can define other geometric structures on ξ like metric, connection, etc. – they have to be defined in every coordinate bundle ξ_α and compatible with the transition isomorphisms $\phi_{\alpha\beta}$.

A common example of gluing is when the manifolds $\widetilde{U}_\alpha$ coincide with U_α, and ξ_α are the trivial bundles $\pi_\alpha : U_\alpha \times \mathbb{R}^n \to U_\alpha$. If $U_\alpha \cap U_\beta \neq \emptyset$ and

$(b,u) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^n$, then $\phi_{\alpha\beta}(b,u) = (b, g_{\alpha\beta}(b)u)$, where $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n,\mathbb{R})$ satisfy the cocycle conditions

$$g_{\alpha\beta}(b)\, g_{\beta\gamma}(b) = g_{\alpha\gamma}(b) \qquad \text{for all } b \in U_\alpha \cap U_\beta \ .$$

2.2. Action of Lie groups on vector bundles

Let G be a Lie group (not necessarily compact) that acts from the left on the vector bundle $\xi = (E, \pi, B)$ by vector bundle morphisms. We denote this action by (T, t), where

$$T : G \times E \to E \ , \qquad t : G \times B \to B \ , \tag{2.7}$$

so that for each $g \in G$ the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "T_g"] \arrow[d, "\pi"'] & E \arrow[d, "\pi"] \\ B \arrow[r, "t_g"] & B \end{tikzcd} \tag{2.8}$$

(i.e., t_g is the action of T_g on E projected to the base B), and $T_g : \xi_b \to \xi_{t_g(b)}$ is a linear isomorphism for any $b \in B$.

The action (T, t) of G on ξ induces a natural action of G on the set $C^\infty(\xi)$ of all sections of ξ by

$$g(\psi) = T_g \circ \psi \circ t_g^{-1} \ , \qquad g \in G \ . \tag{2.9}$$

If we think of $C^\infty(\xi)$ as a vector space (where the multiplication by a number and the addition are defined pointwise), then (2.9) determines an infinite-dimensional linear representation of G. We say that a section $\psi \in C^\infty(\xi)$ is *G-invariant* or *G-equivariant* if $g(\psi) = \psi$, or, in more detail,

$$\psi(t_g(b)) = T_g(\psi(b)) \qquad \text{for all } g \in G \text{ and } b \in B \ . \tag{2.10}$$

It would be more precise to say that a section is invariant with respect to the action (T, t) of G, but we often use "G-invariant" for brevity.

Let $C^\infty(\xi)^G$ stand for the set of all G-invariant sections of ξ. Clearly, $C^\infty(\xi)^G$ is a linear subspace of $C^\infty(\xi)$. Obviously, the problem of complete description of $C^\infty(\xi)^G$ for a general smooth action of a Lie group G is very complicated and little can be said. Even in the degenerate case when B consists of only one point b and T is a representation of G in the only fiber ξ_b, the problem is not easy – it reduces to description of all invariants of the finite-dimensional representations of G.

Remark 2.1. If (x^μ, z^a) are local coordinates in ξ, the action (T_g, t_g) of $g \in G$ has the form

$$T_g(x^\mu, z^a) = (t_g(x)^\mu, u^a{}_b(g,x)\, z^b) \ ,$$

where $u(g,x) \in GL(n,\mathbb{R})$ satisfies $u(g_1 g_2, x) = u(g_1, t_{g_2}(x))\, u(g_2, x)$ for all $g_1, g_2 \in G$. The action (2.9) of G on the sections $\psi \in C^\infty(\xi)$ becomes

$$g(\psi)^a(x) = u^a{}_b(g, t_{g^{-1}}(x))\, \psi^b(t_{g^{-1}}(x)) \ ,$$

and the invariance condition (2.10) reads $\psi^a(t_g(x)) = u^a{}_b(g,x)\, \psi^b(x)$.

To simplify the problem of the description of $C^\infty(\xi)^G$, we will impose two conditions which will be assumed to hold throughout the rest of the paper.

2.2.1. Condition A (on the action of G on B). The first condition is on the action t of the Lie group G on the base B of the vector bundle ξ.

Condition A. *All orbits of the action $t : G \times B \to B$ of G on the base B of ξ are of the same type, and the quotient space B/G is a manifold. Moreover, the orbits of the action t form a locally trivial G-bundle (B, p, N), where*

$$p : B \to N := B/G \tag{2.11}$$

is the natural projection.

Let us discuss Condition A in more detail. First, note that N is just a short notation for the base B/G of the bundle (2.11). The manifold N – which will be the base of the reduced bundle in Section 2.3 – does not have a canonical realization, so one of the tasks in constructing the reduced bundle will be to glue N out of some concrete submanifolds of B.

To fully understand the meaning of Condition A, we recall some facts (for more details, see [9, Chapters 1 and 2]). Let $H \subseteq G$ be a closed subgroup of G, and G/H be the space of left cosets of H in G; the canonical left action of G on G/H is $(g, [g_1]) \mapsto g[g_1] := [gg_1]$. The group of G-equivariant automorphisms of G/H,

$$\mathrm{Aut}^G(G/H) = \{\, f : G/H \to G/H \;:\; f(g[g_1]) = gf([g_1]) \,\} ,$$

is isomorphic to $\mathcal{N}(H)/H$, where $\mathcal{N}(H) = \{\, g \in G \;:\; gHg^{-1} \subseteq H \,\}$ is the normalizer of H in G. The isomorphism $\mathcal{N}(H)/H \to \mathrm{Aut}^G(G/H)$ is given by

$$\mathcal{N}(H)/H \to \mathrm{Aut}^G(G/H) : [n] \mapsto f_{[n]} , \qquad f_{[n]}([g]) := [gn^{-1}] . \tag{2.12}$$

Condition A requires that there exist a closed subgroup H of G such that all orbits of the action t of G on B are homogeneous G-spaces that are G-isomorphic to G/H (in the sense that, for any $x \in N$, if by this isomorphism $b \mapsto [g_1]$, then $t_g(b) \mapsto g[g_1]$ for any $g \in G$). Moreover, Condition A demands that (B, p, N) be a G-bundle that is locally trivial in the following sense: each point $x \in N$ has a neighborhood $U \subseteq N$ for which there exists a diffeomorphism $\Phi : p^{-1}(U) \to U \times (G/H)$ satisfying

$$\Phi(t_g(b)) = (p(b), g(\pi_2 \circ \Phi(b))) , \tag{2.13}$$

where $b \in p^{-1}(U)$, $g \in G$, and $\pi_2 : U \times (G/H) \to G/H : (v, [g_1]) \mapsto [g_1]$ is the canonical projection. If $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ is a fine enough cover of N so that the diffeomorphisms $\Phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times (G/H)$ satisfy (2.13) for each $\alpha \in \mathscr{A}$, then on $U_\alpha \cap U_\beta \neq \emptyset$ the following condition holds:

$$\Phi_\alpha \circ \Phi_\beta^{-1}(x, [g]) = (x, \phi_{\alpha\beta}(x)([g])) ,$$

where $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Aut}^G(G/H)$ are the transition isomorphisms. The bundle (2.11) can be defined as a bundle with fiber the homogeneous space G/H and transition isomorphisms $\phi_{\alpha\beta}(x) \in \mathrm{Aut}^G(G/H)$, $x \in U_\alpha \cap U_\beta \neq \emptyset$.

Remark 2.2. Via the isomorphism (2.12), $\phi_{\alpha\beta}(x)$ can be considered as an element of the group $\mathcal{N}(H)/H$ as well as a transformation $\mathcal{N}(H)/H \to \mathcal{N}(H)/H$ defined as a left multiplication: $(\phi_{\alpha\beta}(x),[n]) \mapsto \phi_{\alpha\beta}(x)[n]$. Because of this, the transition functions $\phi_{\alpha\beta}$ define a principal bundle $P(\mathcal{N}(H)/H, N)$ with structure group $\mathcal{N}(H)/H$ and base N. Then the bundle (2.11) is associated with the principal bundle $P(\mathcal{N}(H)/H, N)$ through the action of $\mathcal{N}(H)/H$ on G/H defined by (2.12).

Remark 2.3. Condition A is satisfied when G is compact according to the Slice Theorem [9, Section 4.4].

Another result showing that Condition A is commonly encountered is the Principal Orbits Theorem [9, Theorem 4.27]. This theorem states that if G is a compact Lie group and B/G is connected, then there exists a maximum orbit type, G/H, in B, and the union of all orbits of type G/H (called *principal orbits*) is open and dense in B.

Remark 2.4. Here we define some objects that will be needed in the construction of the reduced bundle in Section 2.3. A diffeomorphism $f : B \to B$ is said to be *vertical* (with respect to the projection p in (2.11)) if $p \circ f = p$. Let $\mathrm{Diff}^{\mathrm{v}}(B)$ stand for the group of all vertical diffeomorphisms.

The group of *local actions* of G is defined as follows:

$$G_{\text{loc. act.}} = \{ f \in \mathrm{Diff}^{\mathrm{v}}(B) \,:\, \forall\, x \in N \;\exists\, g_x \in G \text{ s.t. } f(b) = t_{g_x}(b),\ \forall\, b \in p^{-1}(x) \}$$

(where $N = B/G$ as in (2.11)). In other words, the restriction of $f \in G_{\text{loc. act.}}$ to each fiber $p^{-1}(x)$ coincides with the action of some element $g_x \in G$. Each map $\chi : N \to G$ defines an element $f_\chi \in G_{\text{loc. act.}}$ by

$$f_\chi(b) = t_{\chi(p(b))}(b) \ , \qquad b \in B \ . \tag{2.14}$$

The elements of $G_{\text{loc. act.}}$ are in a natural bijective correspondence with the sections of a bundle of groups with fiber $\mathcal{N}(H)/H$, associated with the principal bundle $P(\mathcal{N}(H)/H, N)$ via the action Inn of $\mathcal{N}(H)/H$ on $\mathcal{N}(H)/H$ by inner automorphisms: $\mathrm{Inn}_{[n]}([n_1]) = [n n_1 n^{-1}]$.

Remark 2.5. The paper [10] considers in detail some topological questions related to Condition A.

2.2.2. Condition B (on the action of G on ξ). This condition concerns the action $T : G \times E \to E$ of G on the total space of ξ. For each point $x \in N$, $p^{-1}(x) \subseteq B$ is an orbit of the action t, and, hence, is a homogeneous G-space which is G-isomorphic to G/H for some closed subgroup $H \subseteq G$. In Condition B we want to impose restrictions on the structure of the vector bundles $\xi_{p^{-1}(x)}$, $x \in N$. Firstly, we will require that all G-bundles $\xi_{p^{-1}(x)}$, $x \in N$, be G-isomorphic to some "typical" G-bundle $\zeta := (E', \pi', G/H)$ over G/H endowed with a left action (T', t') of G. Let U be a small enough open subset of N, and $\xi_{p^{-1}(U)} := \xi\restriction_{p^{-1}(U)} = \pi^{-1} \circ p^{-1}(U)$. Since $p^{-1}(U)$ is foliated by the orbits of the action t, we demand that the G-vector bundle $\xi_{p^{-1}(U)}$ be modeled after the direct product $U \times \zeta$. In more detail, let

$$U \times \zeta := (U \times E',\ \mathrm{Id} \times \pi',\ U \times (G/H)) \tag{2.15}$$

be a bundle with projection

$$(\mathrm{Id} \times \pi')(v, w) = (v, \pi'(w)) \qquad \text{for } (v, w) \in U \times E' \ .$$

The bundle $U \times \zeta$ is a G-vector bundle with action (T'', t'') of G defined by

$$T''_g(v, w) = (v, T'_g(w)) \qquad \text{for } (v, w) \in U \times E' \ .$$

Now Condition B can be stated as follows.

Condition B. *We assume that all G-vector bundles $\xi_{p^{-1}(x)}$, $x \in N$, are isomorphic to one another and to the "typical" G-vector bundle $\zeta = (E', \pi', G/H)$ endowed with a left action (T', t') of G.*

The collection of vector bundles $\left\{\xi_{p^{-1}(x)}\right\}_{x \in N}$ forms a locally trivial G-vector bundle over N in the sense that each point $x \in N$ has a neighborhood $U \subseteq N$ such that $\xi_{p^{-1}(U)} = \pi^{-1} \circ p^{-1}(U)$ is isomorphic as a G-vector bundle to $U \times \zeta$ (2.15). In other words, there exists a vector bundle isomorphism $\Psi : \pi^{-1} \circ p^{-1}(U) \to U \times E'$ satisfying

$$\Psi(T_g(e)) = (x, T'_g(w)) \ ,$$

where $e \in \pi^{-1} \circ p^{-1}(U)$ and $\Psi(e) = (x, w) \in U \times E'$ with $x = p \circ \pi(e)$.

Remark 2.6. Condition B implies Condition A, but we considered Condition A independently because of its importance.

2.2.3. Reducible vector bundles.

Definition 2.7. For $b \in B$, let

$$G_b := \{ \, g \in G \, : \, t_g(b) = b \, \} \subseteq G$$

be the *stationary* (or *isotropy*) *group* of b with respect to the action t of G on B. Define the *stationary subspace* of ξ_b (with respect to the linear representation T of G_b in the fiber ξ_b) as

$$\mathrm{st}\, \xi_b := \{ \, u \in \xi_b \, : \, T_g(u) = u \ \ \forall \, g \in G_b \, \} \subseteq \xi_b \ .$$

Condition B guarantees that the family of vector spaces $\mathrm{st}\, \xi_b \subseteq \xi_b$ form a smooth vector subbundle of ξ which we denote by $\mathrm{st}\, \xi$ and call the *stationary subbundle* of ξ (with respect to the action (T, t) of G on ξ).

Definition 2.8. We say that the vector bundle ξ with action (T, t) of G on it is a *reducible G-vector bundle* if the actions t and T satisfy Conditions A and B.

2.3. Reduced vector bundles: construction

Let ξ be a reducible G-vector bundle, i.e., the action (T, t) (2.7) of G on ξ satisfies Conditions A and B from Section 2.2. Then the space $C^\infty(\xi)^G$ of all G-invariant sections of ξ has the structure of the space of *all* sections in some bundle ξ^G which we will call the *reduced vector bundle*. In other words, one can construct and a natural bijective correspondence

$$\theta : C^\infty(\xi^G) \to C^\infty(\xi)^G \tag{2.16}$$

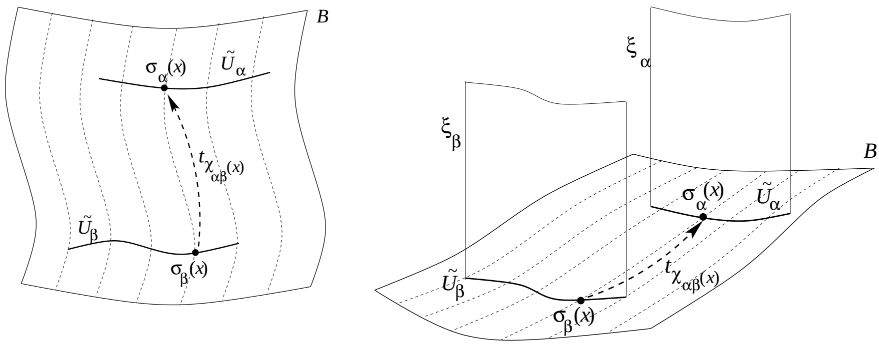

FIGURE 1. *Left:* Constructing the base $N = B/G$ of the reduced bundle ξ^G by gluing it from the manifolds $\{\widetilde{U}_\alpha\}_{\alpha\in\mathscr{A}}$; $\widetilde{U}_\alpha = \sigma_\alpha(U_\alpha)$ is transversal to the orbits of t_g (the orbits of t_g are drawn with dashed lines). *Right:* Constructing the reduced bundle ξ^G by gluing it from the coordinate bundles $\{\xi_\alpha\}_{\alpha\in\mathscr{A}}$; each ξ_α is a vector bundle with base $\widetilde{U}_\alpha$ and fiber $(\xi_\alpha)_{\sigma_\alpha(x)} = \operatorname{st}\xi_{\sigma_\alpha(x)}$ over $\sigma_\alpha(x) \in \widetilde{U}_\alpha$.

between all sections of ξ^G and all G-invariant sections of ξ. Below we describe the explicit construction of ξ^G and θ.

We start with an explicit construction of the base $N = B/G$ of the reduced bundle by gluing it (as explained in Section 2.1) from explicitly defined submanifolds of B. Let $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ be a fine enough open cover of N. For each $\alpha \in \mathscr{A}$ we choose a (smooth) local section σ_α of the bundle (2.11) whose graph

$$\widetilde{U}_\alpha := \sigma_\alpha(U_\alpha) \tag{2.17}$$

is transversal to the fibers of the bundle (2.11) (recall that the fibers of (2.11) are the orbits of the action t of G on B). Clearly, $\sigma_\alpha : U_\alpha \to \widetilde{U}_\alpha$ are diffeomorphisms, and the maps $\sigma_\alpha \circ \sigma_\beta^{-1} : \widetilde{U}_{\beta,\alpha} \to \widetilde{U}_{\alpha,\beta}$ glue the manifold N from the manifolds $\{\widetilde{U}_\alpha\}_{\alpha\in\mathscr{A}}$; here, as before, $\widetilde{U}_{\alpha,\beta} := \sigma_\alpha(U_\alpha \cap U_\beta)$ (cf. (2.1) and (2.2)). This construction is pictorially represented in the left part of Fig. 1.

Now we will construct the coordinate bundles ξ_α (over $\widetilde{U}_\alpha$) of the reduced bundle ξ^G. First of all, it is easy to see from (2.10) that if ψ is a G-invariant section of ξ, then $\psi(b) \in \operatorname{st}\xi_b$, so that $C^\infty(\xi)^G$ is a subset of $C^\infty(\operatorname{st}\xi)$. We define the coordinate bundles

$$\xi_\alpha := \operatorname{st}\xi\restriction_{\widetilde{U}_\alpha} \tag{2.18}$$

as the restrictions of the base of the stationary bundle $\operatorname{st}\xi$ to the manifolds $\widetilde{U}_\alpha$; see the right part of Fig. 1.

The isomorphisms $\phi_{\alpha\beta}$ (2.3) gluing the family $\{\xi_\alpha\}_{\alpha\in\mathscr{A}}$ into the reduced bundle ξ^G are constructed as follows. Let (α, β) be a pair of indices for which

$U_\alpha \cap U_\beta \neq \emptyset$, and define $\xi_{\alpha,\beta} := \mathrm{st}\,\xi\restriction_{\widetilde{U}_{\alpha,\beta}}$. Let

$$\chi_{\alpha\beta} : U_\alpha \cap U_\beta \to G \tag{2.19}$$

be a map that satisfies

$$t_{\chi_{\alpha\beta}(x)}\big(\sigma_\beta(x)\big) = \sigma_\alpha(x) \qquad \text{for all } x \in U_\alpha \cap U_\beta \ , \tag{2.20}$$

as shown in Fig. 1. Similarly to (2.14), define the local action $f_{\chi_{\alpha\beta}}$ of G on $p^{-1}(U_\alpha \cap U_\beta)$ by

$$f_{\chi_{\alpha\beta}} : p^{-1}(U_\alpha \cap U_\beta) \to p^{-1}(U_\alpha \cap U_\beta) : b \mapsto f_{\chi_{\alpha\beta}}(b) := t_{\chi_{\alpha\beta}(p(b))}(b) \ . \tag{2.21}$$

The requirement (2.20) guarantees that $f_{\chi_{\alpha\beta}}(\widetilde{U}_{\beta,\alpha}) = \widetilde{U}_{\alpha,\beta}$.

Next, define the action $F_{\chi_{\alpha\beta}}$ of G on $\pi^{-1}\left(p^{-1}(U_\alpha \cap U_\beta)\right)$ by

$$\begin{aligned} F_{\chi_{\alpha\beta}} &: \pi^{-1}\left(p^{-1}(U_\alpha \cap U_\beta)\right) \to \pi^{-1}\left(p^{-1}(U_\alpha \cap U_\beta)\right) \\ &: e \mapsto F_{\chi_{\alpha\beta}}(e) := T_{\chi_{\alpha\beta}(p(\pi(e)))}(e) \ . \end{aligned} \tag{2.22}$$

Clearly, the actions $f_{\chi_{\alpha\beta}}$ and $F_{\chi_{\alpha\beta}}$ are compatible: $\pi \circ F_{\chi_{\alpha\beta}} = f_{\chi_{\alpha\beta}} \circ \pi$. Therefore the pair $\left(F_{\chi_{\alpha\beta}}, f_{\chi_{\alpha\beta}}\right)$ defines an isomorphism $\phi_{\alpha\beta} : \xi_{\beta,\alpha} \to \xi_{\alpha,\beta}$: if $e \in \xi_{\beta,\alpha} = \mathrm{st}\,\xi\restriction_{\widetilde{U}_{\beta,\alpha}}$ and $x = p \circ \pi(e) \in U_\alpha \cap U_\beta$, then

$$\phi_{\alpha\beta}(e) = T_{\chi_{\alpha\beta}(x)}(e) \in \mathrm{st}\,\xi_{\sigma_\alpha(x)} \ . \tag{2.23}$$

The isomorphism $\phi_{\alpha\beta}$ does not depend on the arbitrariness in the choice of $\chi_{\alpha\beta}$ in (2.19) and is uniquely defined for each pair of indices (α, β) for which $U_\alpha \cap U_\beta \neq \emptyset$. Indeed, if $\chi'_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ is another map satisfying (2.20), then $T_{\chi'_{\alpha\beta}(x)} = T_{\chi_{\alpha\beta}(x)} \circ T_{\chi^{-1}_{\alpha\beta}(x)\,\chi'_{\alpha\beta}(x)}$, but $\chi^{-1}_{\alpha\beta}(x)\,\chi'_{\alpha\beta}(x)$ belongs to the stationary group $G_{\sigma_\beta(x)}$, hence the operator $T_{\chi^{-1}_{\alpha\beta}(x)\,\chi'_{\alpha\beta}(x)}$ is the identity when acting on $\mathrm{st}\,\xi_{\sigma_\beta(x)}$. It is easy to check that isomorphisms $\phi_{\alpha\beta}$ satisfy the cocycle conditions (2.4), hence they glue the reduced bundle ξ^G from the coordinate bundles $\{\xi_\alpha\}_{\alpha \in \mathscr{A}}$.

The construction of ξ^G makes the correspondence (2.16) explicit. A section $S \in C^\infty(\xi^G)$ of the reduced bundle corresponds to a family of sections $S_\alpha \in C^\infty(\xi_\alpha)$ compatible with the transition isomorphisms $\phi_{\alpha\beta}$:

$$S_\alpha = \phi_{\alpha\beta}(S_\beta) \tag{2.24}$$

(using the notation of (2.6)). There exists a unique G-invariant section $\psi = \theta(S) \in C^\infty(\xi)^G$ whose values over $\widetilde{U}_\alpha$ coincide with the values of S_α, i.e., $\psi(\sigma_\alpha(x)) = S_\alpha(\sigma_\alpha(x))$ for all $\alpha \in \mathscr{A}$ and $x \in \widetilde{U}_\alpha$. Namely, for each $b \in B$, we define

$$\psi(b) := T_g(S_\alpha(\sigma_\alpha(x))) \ , \tag{2.25}$$

where $x = p(b) \in U_\alpha$ for some $\alpha \in \mathscr{A}$, and $g \in G$ is such that $t_g(\sigma_\alpha(x)) = b$. Since $S_\alpha(x) \in \mathrm{st}\,\xi_{\sigma_\alpha(x)}$, the value of $\psi(b)$ does not depend on the arbitrariness in the choice of g; moreover, (2.23) and (2.24) imply that if $x = p(b) \in U_\alpha \cap U_\beta$, the value of $\psi(b)$ obtained as in (2.25) but by using S_β instead of S_α would give the same value.

Conversely, given $\psi \in C^\infty(\xi)^G$, its values belong to the stationary subbundle $\mathrm{st}\,\xi$ and, therefore, the restrictions $S_\alpha \in C^\infty(\mathrm{st}\,\xi\restriction_{\widetilde{U}_\alpha}) = C^\infty(\xi_\alpha)$, defined by

$$S_\alpha(\sigma_\alpha(x)) := \psi(\sigma_\alpha(x)) \ , \tag{2.26}$$

are sections of the coordinate bundles ξ_α. Thanks to the G-invariance of ψ (2.10), S_α satisfy (2.24), so they determine a section $S = \theta^{-1}(\psi) \in C^\infty(\xi^G)$.

We summarize the above construction in the following

Theorem 2.9. *Let ξ be a G-reducible vector bundle, i.e., the action (T,t) of the Lie group G on ξ satisfy Conditions A and B from Section* 2.2*. Then there exists a bijective correspondence θ* (2.16) *between all sections $C^\infty(\xi^G)$ of a vector bundle ξ^G (called the reduced vector bundle) and all G-invariant sections $C^\infty(\xi)^G$ of ξ.*

The base of ξ^G is the quotient space B/G. Let $\{U_\alpha\}_{\alpha\in\mathscr{A}}$ be an open cover of B/G, and, for each $\alpha \in \mathscr{A}$, $\sigma_\alpha : U_\alpha \to B$ be a local section of the bundle (2.11) *whose graph, $\widetilde{U}_\alpha$* (2.17)*, is transversal to the orbits of t in B. Then the restrictions $\mathrm{st}\,\xi\restriction_{\widetilde{U}_\alpha}$ are the coordinate bundles ξ_α from which the reduced bundle ξ^G is glued via the isomorphisms $\phi_{\alpha\beta}$* (2.23)*.*

The bijection θ (2.16) *is given explicitly by* (2.24)*,* (2.25)*, and* (2.26)*.*

Remark 2.10. Since each G-invariant function on B, $f \in C^\infty(B)^G$, is constant on each orbit of G in B, the set $C^\infty(\xi)^G$ of all G-invariant sections of ξ is a module over $C^\infty(B)^G$. From the construction of ξ^G above, it is clear that the set of all its sections, $C^\infty(\xi^G)$, is a module over the ring $C^\infty(B/G)$. Then the map θ (2.16) is obviously a homomorphism from the $C^\infty(B/G)$-module $C^\infty(\xi^G)$ to the $C^\infty(B)^G$-module $C^\infty(\xi)^G$.

2.4. Reduced vector bundles: algebraic properties

In this section we will prove several simple lemmata needed in Section 4.2.

Lemma 2.11. *Let ξ_1 and ξ_2 be reducible G-vector bundles over the same base B with the same action t of G on B. Then the Whitney sum $\xi = \xi_1 \oplus \xi_2$ is a reducible G-vector bundle, and*

$$\xi^G = (\xi_1 \oplus \xi_2)^G = \xi_1^G \oplus \xi_2^G \ . \tag{2.27}$$

Proof. If (T_1, t) and (T_2, t) be corresponding actions, then the natural action $(T_1 \oplus T_2, t)$ of G on $\xi_1 \oplus \xi_2$ is defined by

$$(T_1 \oplus T_2)_g(a_1 \oplus a_2) := T_{1,g}(a_1) \oplus T_{2,g}(a_2) \in (\xi_1 \oplus \xi_2)_{t_g(b)} \ ,$$

where $a_1 \oplus a_2 \in (\xi_1 \oplus \xi_2)_b = \xi_{1,b} \oplus \xi_{2,b}$. Clearly, $\mathrm{st}\,(\xi_1 \oplus \xi_2) = \mathrm{st}\,\xi_1 \oplus \mathrm{st}\,\xi_2$, so the coordinate bundles of $(\xi_1 \oplus \xi_2)^G$ have the form $(\xi_1 \oplus \xi_2)_\alpha = \xi_{1,\alpha} \oplus \xi_{2,\alpha}$. The transition isomorphisms $\phi_{\alpha\beta}$ – constructed through the action of G on ξ by (2.19), (2.20), (2.23) – preserve the direct sum and, therefore, endow the reduced bundle ξ^G with the structure of a Whitney sum of ξ_1^G and ξ_2^G. □

Lemma 2.12. *Let ξ_1 and ξ_2 be reducible G-vector bundles over the same base B with the same action t of G on B, and let the action t be free. Then the tensor product $\xi = \xi_1 \otimes \xi_2$ is a reducible G-vector bundle, and*

$$\xi^G = (\xi_1 \otimes \xi_2)^G = \xi_1^G \otimes \xi_2^G \ . \tag{2.28}$$

Proof. The proof is analogous to the proof of Lemma 2.11. The requirement for the action t to be free guarantees that $\mathrm{st}\,(\xi_1 \otimes \xi_2) = \mathrm{st}\,\xi_1 \otimes \mathrm{st}\,\xi_2$ because the fact that G_b is trivial for all $b \in B$ guarantees that $\mathrm{st}\,\xi_1 = \xi_1$ and $\mathrm{st}\,\xi_2 = \xi_2$. □

In general, if the action t of G on B is not free, one can only claim that $\mathrm{st}\,\xi_1 \otimes \mathrm{st}\,\xi_2 \subseteq \mathrm{st}\,(\xi_1 \otimes \xi_2)$. The simplest example to keep in mind is when the common base of the vector bundles ξ_1 and ξ_2 is a single point, the group G is the multiplicative group of all non-zero numbers, and its actions T_1 and T_2 on ξ_1 and ξ_2 are given respectively by $T_{1,g}(a_1) := g \cdot a_1$ and $T_{2,g}(a_2) := \frac{1}{g} \cdot a_2$ (where $g \in \mathbb{R} \setminus \{0\}$, and the dot stands for multiplication). Then, clearly, $\mathrm{st}\,\xi_1$ and $\mathrm{st}\,\xi_2$ consist of the zero vectors only, while $\mathrm{st}\,(\xi_1 \otimes \xi_2) = \xi_1 \otimes \xi_2$.

If the action t of G on B is not free, the representation of G_b in the fiber $\mathrm{st}\,\xi_{1,b} \otimes \mathrm{st}\,\xi_{2,b}$ is a tensor product of the representations T_1 and T_2, and finding all vectors fixed with respect to this representation becomes a problem of Clebsch-Gordan type of finding all stationary vectors in $(\mathrm{st}\,(\xi_1 \otimes \xi_2)_b$.

Another useful lemma is the following.

Lemma 2.13. *Let ξ_1 and ξ_2 be reducible G-vector bundles over the same base B and with the same action t of G on B. Let the action of G on ξ_2 be such that $\mathrm{st}\,\xi_2 = \xi_2$. Then $\xi = \xi_1 \otimes \xi_2$ is a reducible G-vector bundle, and*

$$\xi^G = (\xi_1 \otimes \xi_2)^G = \xi_1^G \otimes \xi_2^G \ . \tag{2.29}$$

2.5. Dimensional reduction of a group action

Let $\xi = (E, \pi, B)$ be a reducible G-vector bundle on which another Lie group K acts by vector bundle morphisms $F : K \times E \to E$, $f : K \times B \to B$, i.e., $\pi \circ F_k = f_k \circ \pi$ for any $k \in K$, and $F_k : \xi_b \to \xi_{f_k(b)}$ is a linear isomorphism for any $b \in B$. Since the actions of G and K commute, the action of K maps G-invariant sections of ξ into G-invariant sections. In other words, $C^\infty(\xi)^G$ is a K-invariant subset of $C^\infty(\xi)$. Because of this we can define in a natural way a G-reduced action (F^G, f^G) of K on the reduced bundle $\xi^G = (\mathrm{st}\,\xi_N, \pi^G, N)$ as follows. Recall that the base N of the G-reduced bundle is glued from submanifolds of B as explained in Section 2.3. We consider the case of only one global chart $N \subseteq B$ leaving out the details about the case when a global chart does not exists – the general case follows the same ideas, *mutatis mutandis*. Let $k \in K$, $b \in N$, and let $\sigma \in (\xi^G)_b = \mathrm{st}\,\xi_b$ be a vector in the fiber over b that is stationary with respect of action of the stationary group G_b of b. In general, $f_k(b)$ does not necessarily belong to N. Let $g \in G$ be such that $t_g \circ f_k(b) \in N$, and define the G-reduced action (F^G, f^G) of K by

$$(f^G)_k(b) := t_g \circ f_k(b) \in N \ , \qquad (F^G)_k(\sigma) := T_g \circ F_k(\sigma) \in (\xi^G)_{(f^G)_k(b)} \ . \tag{2.30}$$

The action k^G of $k \in K$ on the sections of the reduced bundle, $C^\infty(\xi^G)$, is given by

$$k^G : C^\infty(\xi^G) \to C^\infty(\xi^G) : S \mapsto k^G(S) := \theta^{-1} \circ k \circ \theta \circ S \ ,$$

where the action $k : C^\infty(\xi) \to C^\infty(\xi)$ is defined as $k(\psi) = F_k \circ \psi \circ f_k^{-1}$ (cf. (2.9)), and θ is the bijection (2.16). It is an easy exercise to check that the non-uniqueness of $g \in G$ in (2.30) is immaterial for the above procedure.

3. Cohomology of groups and dimensional reduction of short exact sequences of vector bundles

3.1. Splittings of short exact sequences

In this section we collect some elementary facts concerning short exact sequences of vector spaces and G-modules which will be needed later.

Let

$$0 \longrightarrow L_0 \xrightarrow{\ i\ } L \xrightarrow{\ j\ } L_1 \longrightarrow 0 \tag{3.1}$$

be a short exact sequence of vector spaces (i.e., i and j are linear maps satisfying $\ker i = \{0\}$, $\operatorname{im} j = L_1$, and $\operatorname{im} i = \ker j$). The middle term L is isomorphic to $L_0 \oplus L_1$, but this isomorphism is not defined naturally. The choice of such an isomorphism – called a *splitting* of the short exact sequence (3.1) – is equivalent to defining a subspace of L that is transversal to $i(L_0) \subseteq L$. This can be achieved by specifying a linear map $S \in \mathrm{Hom}\,(L_1, L)$ satisfying $j \circ S = \mathrm{Id}_{L_1}$ or, equivalently, a linear map $F \in \mathrm{Hom}\,(L, L_0)$ that satisfies $F \circ i = \mathrm{Id}_{L_0}$; since we will refer to these properties often, we collect them here:

$$j \circ S = \mathrm{Id}_{L_1} \ , \qquad F \circ i = \mathrm{Id}_{L_0} \ . \tag{3.2}$$

If (3.2) are satisfied, $\ker F$ and $\operatorname{im} S$ are transversal to $i(L_0)$. The isomorphism $L \cong L_0 \oplus L_1$ is given by the pair of embeddings (i, S) of L_0 and L_1 into L or, equivalently, by the pair of projections (F, j) from L onto L_0 and L_1; the isomorphism is given by $L = i(L_0) \oplus S(L_1) = \ker j \oplus \ker F$. Clearly, the maps S and F define the same splitting of (3.1) if and only if $\ker F = \operatorname{im} S$; in this case, if S is known, the map F is given by $F = i^{-1} \circ (\mathrm{Id} - S \circ j)$. On the other hand, given S, we have $S(v_1) = (\mathrm{Id} - i \circ F)(v)$, where $v \in L$ is such that $j(v) = v_1$. Although giving either S or F defines a splitting completely, for convenience we will often say "splitting (F, S)" meaning that F and S are a pair of maps corresponding to the same splitting. A splitting (F, S) of (3.1) gives us the following decomposition of the identity in L as a sum of projection operators onto the subspaces $i(L_0)$ and $S(L_1)$:

$$\mathrm{Id}_L = i \circ F + S \circ j \ . \tag{3.3}$$

If S_1, S_2, F_1, and F_2 define splittings of (3.1), then

$$S_2 = S_1 + i \circ s \ , \qquad F_2 = F_1 + f \circ j \ , \tag{3.4}$$

where s and f are linear maps from L_1 to L_0. In other words, the set of all splittings of the short exact sequence (3.1) is an affine space with linear group $\mathrm{Hom}\,(L_1, L_0)$. One can easily see that if (S_1, F_1) and (S_2, F_2) are two splittings, then the maps s and f in (3.4) are related by $s = -f$.

Let D_0, D and D_1 be representations of the group G in L_0, L and L_1, respectively, and let the operators i and j in (3.1) be intertwining, i.e.,

$$D(g) \circ i = i \circ D_0(g) \ , \qquad j \circ D(g) = D_1(g) \circ j \qquad \text{for all } g \in G \ . \tag{3.5}$$

In this case we will say that (3.1) is an *intertwining* short exact sequence of G-modules (in another terminology, the short exact sequence (3.1) is G-equivariant).

If the short exact sequence (3.1) is intertwining, then the subspace $i(L_0)$ is invariant with respect of the representation D; the subspace $S(L_1)$, however, is not D-invariant in general.

For a splitting (F, S) we can use (3.2), (3.5), and the exactness of (3.1) to obtain

$$\begin{aligned} D(g) &= (i \circ F + S \circ j) \circ D(g) \circ (i \circ F + S \circ j) \\ &= i \circ D_0(g) \circ F + i \circ F \circ D(g) \circ S \circ j + S \circ D_1(g) \circ j \ . \end{aligned}$$

Each vector $v \in L$ can be represented as a sum of a vector in $i(L_0)$ and a vector in $S(L_1)$: $v = i(v_0) + S(v_1)$, where $v_0 = F(v) \in L_0$ and $v_1 = j(v) \in L_1$. If we write this symbolically as $v = (v_0\ v_1)^T$, we can write the operator $D(g) \in \mathrm{End}\, L$ as

$$D(g) \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} D_{00}(g) & D_{01}(g) \\ 0 & D_{11}(g) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} ,$$

where we have introduced the “components” of the representation D:

$$\begin{aligned} D_{00}(g) &= F \circ D(g) \circ i = D_0(g) \ , \\ D_{01}(g) &= F \circ D(g) \circ S \in \mathrm{Hom}\,(L_1, L_0) \ , \\ D_{11}(g) &= j \circ D(g) \circ S = D_1(g) \ . \end{aligned}$$

Note that $D_{10}(g) := j \circ D(g) \circ i = 0$ for all $g \in G$. The operator $D_{01}(g)$ satisfies the relation

$$D_{01}(g_1 g_2) = D_{00}(g_1)\, D_{01}(g_2) + D_{01}(g_1)\, D_{11}(g_2) \qquad \text{for all } g_1, g_2 \in G \ .$$

3.2. Facts from cohomology of groups

In this section we collect some facts from cohomology of groups needed for the procedure of dimensional reduction. For details we refer the reader to [11] or, for physics-motivated exposition, to [12].

Definition 3.1. Let G be a topological group and D be a continuous linear representation of G in the (generally infinite-dimensional) vector space L. The set $C^n(G, L)$ of all *n-chains* consists of the continuous functions

$$c : G \times G \times \cdots \times G \to L$$

(n copies of G), and $C^0(G,L) = L$. The *coboundary operator* $\delta^{(n)} : C^n(G,L) \to C^{n+1}(G,L)$ is defined by

$$\begin{aligned}\big(\delta^{(n)}c\big)(g_1,g_2,\dots,g_{n+1}) := D(g_1)\, c(g_2,\dots,g_{n+1}) \\ + \sum_{i=1}^{n}(-1)^i\, c(g_1,\dots,g_{i-1},\, g_i g_{i+1},\, g_{i+2},\dots,g_{n+1}) \\ + (-1)^{n+1} c(g_1,\dots,g_n)\ .\end{aligned} \tag{3.6}$$

The elements of $\ker\delta^{(n)}$ and $\operatorname{im}\delta^{(n)}$ are called *cocycles* and *coboundaries*, respectively. The nth *cohomology group* is defined as $H^n(G,L) := \ker\delta^{(n)}/\operatorname{im}\delta^{(n-1)}$ for $n \in \mathbb{N}$, and $H^0(G,L) := \ker\delta^{(0)}$.

The fact that the cohomology groups are well-defined is based on the fact (proved, e.g., in [12, Theorem 5.1.1]) that the coboundary operator satisfies

$$\delta^{(n+1)} \circ \delta^{(n)} = 0\ .$$

Here are the explicit expressions for $n = 0$ and $n = 1$:

$$\begin{aligned}\big(\delta^{(0)}c\big)(g) &= D(g)c - c\ ,\\ \big(\delta^{(1)}c\big)(g_1,g_2) &= D(g_1)\, c(g_2) - c(g_1\cdot g_2) + c(g_1)\ .\end{aligned}$$

If $c \in H^0(G,L)$, from $\delta^{(0)}c = 0$ we obtain $D(g)c - c = 0$ for all $g \in G$, which means that $c \in \operatorname{st} L$. This leads us to the important observation that

$$H^0(G,L) = \operatorname{st} L\ . \tag{3.7}$$

Now we return to the problem of splitting short exact sequences. Let (F,S) be a splitting of the intertwining short exact sequence (3.1). Define the map $k_n : C^n(G,L_1) \to C^{n+1}(G,L_0)$ by

$$\begin{aligned}(k_n c)(g_1,\dots,g_{n+1}) &:= F\circ D(g_1)\circ S\,(c(g_2,\dots,g_{n+1}))\\ &= D_{01}(g_1)\, c\,(g_2,\dots,g_{n+1})\ .\end{aligned} \tag{3.8}$$

Lemma 3.2. *The map* k_n (3.8) *anticommutes with the coboundary operator in the sense that*

$$\delta^{(n+1)}_{L_0} \circ k_n = -k_{n+1}\circ\delta^{(n)}_{L_1}\ , \tag{3.9}$$

and, hence, defines a map between the cohomology groups

$$k_n : H^n(G,L_1) \to H^{n+1}(G,L_0) \tag{3.10}$$

(for which we use the same notation as for the map (3.8)*). The map* k_n *in* (3.10) *does not depend on the splitting* (F,S) *of the short exact sequence* (3.1).

Proof. The basic step in the proof of (3.9) is the following observation (based on the decomposition (3.3) and the identities (3.2) and (3.5)):

$$\begin{aligned}F\circ D(g_1g_2)\circ S &= F\circ D(g_1)\circ D(g_2)\circ S\\ &= F\circ D(g_1)\circ(i\circ F + S\circ j)\circ D(g_2)\circ S\\ &= D_0(g_1)\circ F\circ D(g_2)\circ S + F\circ D(g_1)\circ S\circ D_1(g_2)\circ S\ .\end{aligned}$$

Using this to rewrite the term underlined in the equations below, we obtain for $c \in C^n(G, L_1)$

$$
\begin{aligned}
\big(\delta_{L_0}^{(n+1)} \circ k_n(c)\big)&(g_1, \dots, g_{n+2}) \\
&= D_0(g_1)(k_n c)(g_2, \dots, g_{n+2}) \\
&\qquad + \sum_{i=1}^{n+1} (-1)^i (k_n c)(g_1, \dots, g_i g_{i+1}, \dots, g_{n+2}) \\
&\qquad + (-1)^{n+2} (k_n c)(g_1, \dots, g_{n+1}) \\
&= D_0(g_1) \circ F \circ D(g_2) \circ S(c(g_3, \dots, g_{n+2})) \\
&\qquad - \underline{F \circ D(g_1 g_2) \circ S}(c(g_3, \dots, g_{n+2})) \\
&\qquad + \sum_{i=2}^{n+1} (-1)^i F \circ D(g_1) \circ S(c(g_2, \dots, g_i g_{i+1}, \dots, g_{n+2})) \\
&\qquad + (-1)^{n+2} F \circ D(g_1) \circ S(c(g_2, \dots, g_{n+1})) \\
&= -F \circ D(g_1) \circ S \circ D_1(g_2) c(g_3, \dots, g_{n+2}) \\
&\qquad + \sum_{i=2}^{n+1} (-1)^i F \circ D(g_1) \circ S(c(g_2, \dots, g_i g_{i+1}, \dots, g_{n+2})) \\
&\qquad + (-1)^{n+2} F \circ D(g_1) \circ S(c(g_2, \dots, g_{n+1})) \\
&= -F \circ D(g_1) \circ S \Big(D_1(g_2) c(g_3, \dots, g_{n+2}) \\
&\qquad\qquad + \sum_{i=2}^{n+1} (-1)^{i-1} c(g_2, \dots, g_i g_{i+1}, \dots, g_{n+2}) \\
&\qquad\qquad + (-1)^{n+1} c(g_2, \dots, g_{n+1}) \Big) \\
&= -F \circ D(g_1) \circ S \left(\big(\delta_{L_1}^{(n)} c\big)(g_2, \dots, g_{n+2}) \right) \\
&= -\big(k_{n+1} \circ \delta_{L_1}^{(n)}(c)\big)(g_1, \dots, g_{n+2}) \ .
\end{aligned}
$$

It remains to prove that the map (3.10) is independent of the splitting of the intertwining short exact sequence (3.1). Let (F, S) be a splitting of (3.1) and k_n be the corresponding map (3.8). Let $(\widetilde{F}, \widetilde{S})$, where $\widetilde{S} = S + i \circ s$ and $\widetilde{F} = F - s \circ j$ for some $s \in \operatorname{Hom}(L_1, L_0)$, be another splitting of (3.1), and $\widetilde{k}_n$ be the corresponding map (3.8). Directly from the definitions we obtain, for $c \in C^n(G, L_1)$,

$$
\big(\delta_{L_0}^{(n)}(s \circ c) - s \circ (\delta_{L_1}^{(n)} c)\big)(g_1, \dots, g_{n+1}) = \big(D_0(g_1) \circ s - s \circ D_1(g_1)\big)(c(g_2, \dots, g_{n+1})) \, .
$$

Using this fact, the identities (3.2) and (3.5) and the exactness of (3.1), we obtain

$$\begin{aligned}\big((\widetilde{k}_n - k_n)c\big)(g_1,\dots,g_{n+1}) &= \big(D_0(g_1)\circ s - s\circ D_1(g_1)\big)(c(g_2,\dots,g_{n+1})) \\ &= \big(\delta^{(n)}_{L_0}(s\circ c) - s\circ(\delta^{(n)}_{L_1}c)\big)(g_1,\dots,g_{n+1})\ .\end{aligned} \tag{3.11}$$

Now let $c \in C^n(G, L_1)$ with $\delta^{(n)}_{L_1} c = 0$, and let $[c] = c + \delta^{(n-1)}_{L_1}\left(C^{n-1}(G, L_1)\right)$ be its equivalence class in $H^n(G, L_1)$. Then (3.11) implies immediately that $(\widetilde{k}_n - k_n)(c) \in \operatorname{im}\delta^{(n)}_{L_0}$, i.e., that $\widetilde{k}_n([c]) = k_n([c])$. □

In order to formulate the fundamental theorem below, we need the following definition.

Definition 3.3. If D_1 and D_2 are representations of the groups G_1 and G_2 in the vector spaces L_1 and L_2, then a *morphism* between these two representations is defined as a pair (ψ, ϕ), where $\psi : G_1 \to G_2$ is a morphism of groups and $\phi : L_1 \to L_2$ is a linear map satisfying

$$\phi\circ D_1(g) = D_2(\psi(g))\circ\phi \qquad \text{for all } g \in G_1\ . \tag{3.12}$$

If ψ is an isomorphism, then (ψ, ϕ) induces a map $(\psi, \phi)_n : C^n(G_1, L_1) \to C^n(G_2, L_2)$ by

$$((\psi,\phi)_n\, c)\,(g_1,\dots,g_n) := \phi\left(c(\psi^{-1}(g_1),\dots,\psi^{-1}(g_n))\right)\ , \tag{3.13}$$

where $g_1, \dots, g_n \in G_2$. The map $(\psi, \phi)_n$ commutes with the coboundary operator in the sense that $(\psi,\phi)_{n+1}\circ\delta^{(n)}_{L_1} = \delta^{(n)}_{L_2}\circ(\psi,\phi)_n$ (which follows immediately from the definitions) and, thus, determines a map

$$(\psi,\phi)_n : H^n(G_1, L_1) \to H^n(G_2, L_2)\ . \tag{3.14}$$

The conditions (3.5) for the short exact sequence (3.1) to be intertwining are a particular case of (3.12) for $G_1 = G_2 =: G$ and $\psi = \mathrm{Id}_G$. Let

$$\begin{aligned} i_n &:= (\mathrm{Id}, i)_n : H^n(G, L_0) \to H^n(G, L)\ , \\ j_n &:= (\mathrm{Id}, j)_n : H^n(G, L) \to H^n(G, L_1)\ , \\ k_n &:= (\mathrm{Id}, k)_n : H^n(G, L_1) \to H^{n+1}(G, L_0)\ . \end{aligned} \tag{3.15}$$

be the maps induced by the morphisms (Id, i), (Id, j) and (Id, k) between the corresponding representations, as in (3.14).

A classic result in cohomology theory is the following [13, Chapter 1]

Theorem 3.4. *If* (3.1) *is an intertwining with respect to the representations of* G *short exact sequence of vector spaces, then the sequence*

$$0 \to H^0(G, L_0) \xrightarrow{i_0} H^0(G, L) \xrightarrow{j_0} H^0(G, L_1) \xrightarrow{k_0} H^1(G, L_0) \xrightarrow{i_1} H^1(G, L) \xrightarrow{j_1} \cdots \tag{3.16}$$

is exact.

Let D be a representation of the group G in the vector space L. Let for $g \in G$, let Inn_g be the inner automorphism

$$\mathrm{Inn}_g : G \to G : \mathrm{Inn}_g(g_1) := g g_1 g^{-1} \ . \tag{3.17}$$

Then the pair $(\mathrm{Inn}_g, D(g))$ is an automorphism of the representation D (in the sense of the definition (3.12)), and one can construct the maps

$$(\mathrm{Inn}_g, D(g))_n : C^n(G, L) \to C^n(G, L)$$

as in (3.13), and the corresponding maps (3.14) from the cohomology group $H^n(G, L)$ to itself. The following lemma holds:

Lemma 3.5. *The map* $(\mathrm{Inn}_g, D(g))_n : H^n(G, L) \to H^n(G, L)$ *induced by* $(\mathrm{Inn}_g, D(g))$ *is trivial for any* $g \in G$.

Proof. For $g \in G$, define the maps $\Omega_i^n(g) : C^n(G, L) \to C^{n-1}(G, L)$ by

$$\big(\Omega_i^n(g)(c)\big)(g_1, \dots, g_{n-1}) := c(g_1, \dots, g_i, g, \mathrm{Inn}_g(g_{i+1}), \dots, \mathrm{Inn}_g(g_{n-1})) \ ,$$

and

$$\Omega^n(g) := \sum_{i=0}^{n-1} (-1)^i \, \Omega_i^n(g) \ : \ C^n(G, L) \to C^{n-1}(G, L) \ .$$

Then a long but straightforward computation gives that for $c \in C^n(G, L)$,

$$\big(\mathrm{Inn}_g, D(g)\big)(c) - c = \delta^{(n-1)}\big(\Omega^n(g)(c)\big) \ ,$$

from which the Lemma follows. ☐

3.3. Dimensional reduction of short exact sequences of vector bundles

Let $\xi_i := (E_i, \pi_i, B)$ (where i stands for 0, 1, or nothing) be reducible G-vector bundles over the same base B, with the same action t of the Lie group G on B. Let actions (T_i, t) of G on ξ_i be such that the short exact sequence

$$0 \longrightarrow \xi_0 \xrightarrow{\ i\ } \xi \xrightarrow{\ j\ } \xi_1 \longrightarrow 0 \tag{3.18}$$

be intertwining, i.e., the vector bundle morphisms $i : E_0 \to E$ and $j : E \to E_1$ commute with the corresponding actions of G:

$$T_g \circ i = i \circ T_{0,\,g} \ , \qquad j \circ T_g = T_{1,\,g} \circ j \qquad \text{for all } g \in G \ . \tag{3.19}$$

The main problem we will study in this section is whether an exact sequence like (3.18) holds for the reduced bundles ξ_0^G, ξ^G, ξ_1^G. Recalling the construction of the reduced bundles from Section 2.3, it is clear that the restrictions $\xi_i \restriction_{\widetilde{U}_\alpha}$ of the bundles ξ_i to the manifolds $\widetilde{U}_\alpha \subseteq B$ (see (2.17)) constitute an intertwining short exact sequence

$$0 \longrightarrow \xi_0\restriction_{\widetilde{U}_\alpha} \xrightarrow{\ i\ } \xi\restriction_{\widetilde{U}_\alpha} \xrightarrow{\ j\ } \xi_1\restriction_{\widetilde{U}_\alpha} \longrightarrow 0 \ . \tag{3.20}$$

The coordinate bundles $\xi_{i,\alpha}$ are obtained from $\xi_i\restriction_{\widetilde{U}_\alpha}$ by restricting each fiber to the stationary subspace in it: $\xi_{i,\,\alpha} = \mathrm{st}\, \xi_i\restriction_{\widetilde{U}_\alpha}$. Therefore, we have to study the behavior of the short exact sequence (3.20) under the process of restricting the bundles $\xi_i \restriction_{\widetilde{U}_\alpha}$ to their stationary subbundles $\mathrm{st}\, \xi_i \restriction_{\widetilde{U}_\alpha} \subseteq \xi_i \restriction_{\widetilde{U}_\alpha}$. For each $x \in U_\alpha$ we can write the following intertwining short exact sequence of finite-dimensional representations of the stationary group $G_{\sigma_\alpha(x)}$ in the vector spaces $\xi_{i,\,\sigma_\alpha(x)}$

$$0 \longrightarrow \xi_{0,\,\sigma_\alpha(x)} \xrightarrow{\ i\ } \xi_{\sigma_\alpha(x)} \xrightarrow{\ j\ } \xi_{1,\,\sigma_\alpha(x)} \longrightarrow 0\ . \tag{3.21}$$

Now the cohomological interpretation (3.7) of the stationary subspaces plays a crucial role: we can think of the coordinate bundle $\xi_{i,\alpha}$ as a bundle over $\widetilde{U}_\alpha = \sigma_\alpha(U_\alpha)$ with fiber

$$\mathrm{st}\,\xi_{i,\,\sigma_\alpha(x)} = H^0(G_{\sigma_\alpha(x)}, \xi_{i,\,\sigma_\alpha(x)}) \qquad \text{for any } x \in U_\alpha\ .$$

According to Theorem 3.4, when we restrict each bundle in (3.21) to its stationary subbundle, we obtain the long exact sequence of vector spaces

$$0 \to \mathrm{st}\,\xi_{0,\,\sigma_\alpha(x)} \xrightarrow{i_0} \mathrm{st}\,\xi_{\sigma_\alpha(x)} \xrightarrow{j_0} \mathrm{st}\,\xi_{1,\,\sigma_\alpha(x)} \xrightarrow{k_0} H^1(G_{\sigma_\alpha(x)}, \xi_{0,\,\sigma_\alpha(x)}) \xrightarrow{i_1} \cdots\ . \tag{3.22}$$

The disjoint union of $H^1(G_{\sigma_\alpha(x)}, \xi_{0,\,\sigma_\alpha(x)})$ for $x \in U_\alpha$, with an appropriate equivalence relation, constitutes a (possibly infinite-dimensional) vector bundle with base $\widetilde{U}_\alpha$ and fiber $H^1(G_{\sigma_\alpha(x)}, \xi_{0,\,\sigma_\alpha(x)})$; we introduce the short notation $H^1(G, \xi_0)_\alpha$ for this bundle. Similar facts hold for all the terms in (3.22), for which we introduce similar notations. Our goal now is to glue the coordinate bundles $H^n(G, \xi_i)_\alpha$ into a bundle $H^n(G, \xi_i)$ over $N \cong B/G$; to this end, we have to construct transition isomorphisms between $H^n(G, \xi_i)_\alpha$ and $H^n(G, \xi_i)_\beta$ satisfying the cocycle conditions (2.4).

Let $\phi_{i,\,\alpha\beta}$ be the transition isomorphisms gluing the coordinate bundles $\xi_{i,\alpha}$ (2.18) into the reduced bundle ξ_i^G (see (2.19), (2.20), (2.21), (2.22), (2.23)). Here we explain how to use them in a natural way in order to construct isomorphisms $\phi^n_{i,\,\alpha\beta}$ gluing the bundles $H^n(G, \xi_i)$ out of the coordinate bundles $H^n(G, \xi_i)_\alpha$ for all $n \in \mathbb{N}$. Let $(\alpha, \beta) \in \mathscr{A} \times \mathscr{A}$ be a pair of indices for which $U_\alpha \cap U_\beta \neq \emptyset$, and let $x \in U_\alpha \cap U_\beta$. Let $\chi_{\alpha\beta}$ be the map defined in (2.19) and (2.20), and let $f_{\chi_{\alpha\beta}}$ and $F_{\chi_{\alpha\beta}}$ be the maps defined in (2.21), (2.22) and used in the process of gluing the reduced bundle ξ^G from the coordinate bundles ξ_α. Recall that the fiber $\xi_{i,\sigma_\alpha(x)}$ of the coordinate bundle $\xi_{i,\alpha}$ over the point $\sigma_\alpha(x) \in \widetilde{U}_\alpha$ is the fiber $\mathrm{st}\,\xi_{i,\sigma_\alpha(x)}$ of the stationary bundle $\mathrm{st}\,\xi_i$ over $\sigma_\alpha(x)$, and similarly for $\xi_{i,\sigma_\alpha(x)}$. Let

$$T_i^{\sigma_\beta(x)} : G_{\sigma_\beta(x)} \times \xi_{i,\sigma_\beta(x)} \to \xi_{i,\sigma_\beta(x)}\ , \tag{3.23}$$

$$T_i^{\sigma_\alpha(x)} : G_{\sigma_\alpha(x)} \times \xi_{i,\sigma_\alpha(x)} \to \xi_{i,\sigma_\alpha(x)} \tag{3.24}$$

be the representations of the stationary groups $G_{\sigma_\beta(x)}$ and $G_{\sigma_\alpha(x)}$ in these fibers. Recalling Definition 3.3, we see that the maps

$$\mathrm{Inn}_{\chi_{\alpha\beta}(x)} : G_{\sigma_\beta(x)} \to G_{\sigma_\alpha(x)} : h \mapsto \mathrm{Inn}_{\chi_{\alpha\beta}(x)} h = \chi_{\alpha\beta}(x)\, h\, \chi_{\alpha\beta}(x)^{-1}$$

and

$$T_{i,\,\chi_{\alpha\beta}(x)} : \xi_{i,\sigma_\beta(x)} \to \xi_{i,\sigma_\alpha(x)}$$

form a pair $\big(\mathrm{Inn}_{\chi_{\alpha\beta}(x)}, T_{i,\,\chi_{\alpha\beta}(x)}\big)$ that is an isomorphism between the representation (3.23) of $G_{\sigma_\beta(x)}$ in $\xi_{i,\sigma_\beta(x)}$ and the representation (3.24) of $G_{\sigma_\alpha(x)}$

in $\xi_{i,\sigma_\alpha(x)}$:

$$T_{i,\,\chi_{\alpha\beta}(x)} \circ T^{\sigma_\beta(x)}_{i,\,g} = T^{\sigma_\alpha(x)}_{i,\,\mathrm{Inn}_{\chi_{\alpha\beta}(x)}(g)} \circ T_{i,\,\chi_{\alpha\beta}(x)} \qquad \text{for all } g \in G_{\sigma_\beta(x)}$$

(cf. (3.12)). Therefore, the maps $\big(\mathrm{Inn}_{\chi_{\alpha\beta}(x)}, T_{i,\,\chi_{\alpha\beta}(x)}\big)_n$ defined by (3.13) and (3.14) are well-defined maps between the corresponding cohomology groups:

$$\big(\mathrm{Inn}_{\chi_{\alpha\beta}(x)}, T_{i,\,\chi_{\alpha\beta}(x)}\big)_n : H^n\big(G_{\sigma_\beta(x)},\, \xi_{i,\sigma_\beta(x)}\big) \to H^n\big(G_{\sigma_\alpha(x)},\, \xi_{i,\sigma_\alpha(x)}\big)\ , \tag{3.25}$$

or, in the short notation introduced above,

$$\big(\mathrm{Inn}_{\chi_{\alpha\beta}(x)}, T_{i,\,\chi_{\alpha\beta}(x)}\big)_n : H^n(G,\xi_i)_{\beta,\,\sigma_\beta(x)} \to H^n(G,\,\xi_i)_{\alpha,\,\sigma_\alpha(x)}\ .$$

The map (3.25) is defined uniquely, although $\chi_{\alpha\beta}$ is not unique. Recall that $\chi_{\alpha\beta}$ is determined by the condition $t_{\chi_{\alpha\beta}(x)}\big(\sigma_\beta(x)\big) = \sigma_\alpha(x)$, and, therefore, is defined up to a multiplication on the right by $g \in G_{\sigma_\beta(x)}$ and a multiplication on the left by $g' \in G_{\sigma_\alpha(x)}$. Lemma 3.5, however, guarantees that the maps

$$\big(\mathrm{Inn}_g, T_{i,\,g}\big)_n : H^n\big(G_{\sigma_\beta(x)},\, \xi_{i,\sigma_\beta(x)}\big) \to H^n\big(G_{\sigma_\beta(x)},\, \xi_{i,\sigma_\beta(x)}\big)$$

and

$$\big(\mathrm{Inn}_{g'}, T_{i,\,g'}\big)_n : H^n\big(G_{\sigma_\alpha(x)},\, \xi_{i,\sigma_\alpha(x)}\big) \to H^n\big(G_{\sigma_\alpha(x)},\, \xi_{i,\sigma_\alpha(x)}\big)$$

are trivial for any $g \in G_{\sigma_\beta(x)}$ and $g' \in G_{\sigma_\alpha(x)}$. Therefore the map (3.25) does not depend on the arbitrariness in the choice of the map $\chi_{\alpha\beta}$. (In the case $n = 0$, this is simply the fact that the map $T_{i,\chi_{\alpha,\beta}(x)} : \mathrm{st}\,\xi_{i,\sigma_\beta(x)} \to \mathrm{st}\,\xi_{i,\sigma_\alpha(x)}$ is independent of the arbitrariness in the choice of $\chi_{\alpha\beta}$, as explained after equation (2.23).) Letting x vary over $U_\alpha \cap U_\beta$, and recalling that $H^n(G,\xi_i)_{\alpha,\beta}$ stands for the restriction $H^n(G,\xi_i)_\alpha \restriction_{\widetilde{U}_{\alpha,\beta}}$ of $H^n(G,\xi_i)_\alpha$ over the manifold $\widetilde{U}_{\alpha,\beta}$ (2.1), we obtain that the maps $\big(\big(\mathrm{Inn}_{\chi_{\alpha\beta}(x)}, T_{i,\,\chi_{\alpha\beta}(x)}\big)_n,\, t_{\chi_{\alpha\beta}(x)}\big)$ define the desired isomorphisms

$$\phi^n_{i,\,\alpha\beta} : H^n(G,\xi_i)_{\beta,\alpha} \to H^n(G,\xi_i)_{\alpha,\beta}\ .$$

The maps $\phi^n_{i,\,\alpha\beta}$ glue the coordinate bundles $\{H^n(G,\xi_i)_\alpha\}_{\alpha\in\mathscr{A}}$ into the bundle $H^n(G,\xi_i)$. Since $\phi^n_{i,\,\alpha\beta}$ are constructed through the actions of G on ξ_i, they satisfy the cocycle conditions (2.4) required in the gluing procedure.

The only thing that remains to be proved is that the isomorphisms $\phi^n_{i,\,\alpha\beta}$ are compatible with the long exact sequences (3.22) of the coordinate bundles, i.e., that the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{st}\,\xi_{0,\,\beta,\alpha} \arrow[r, "i_0"] \arrow[d, "\phi_{0,\,\alpha\beta}"] & \mathrm{st}\,\xi_{\beta,\alpha} \arrow[r, "j_0"] \arrow[d, "\phi_{\alpha\beta}"] & \mathrm{st}\,\xi_{1,\,\beta,\alpha} \arrow[r, "k_0"] \arrow[d, "\phi_{1,\,\alpha\beta}"] & H^1(G,\xi_0)_{\beta,\alpha} \arrow[r, "i_1"] \arrow[d, "\phi^1_{0,\,\alpha\beta}"] & H^1(G,\xi)_{\beta,\alpha} \arrow[r, "j_1"] \arrow[d, "\phi^1_{\alpha\beta}"] & \cdots \\
0 \arrow[r] & \mathrm{st}\,\xi_{0,\,\alpha,\beta} \arrow[r, "i_0"] & \mathrm{st}\,\xi_{\alpha,\beta} \arrow[r, "j_0"] & \mathrm{st}\,\xi_{1,\,\alpha,\beta} \arrow[r, "k_0"] & H^1(G,\xi_0)_{\alpha,\beta} \arrow[r, "i_1"] & H^1(G,\xi)_{\alpha,\beta} \arrow[r, "j_1"] & \cdots
\end{tikzcd} \tag{3.26}$$

is commutative. To give an idea of how the proof goes, let us check a part of the diagram including the connecting morphism k_n. Consider the two short exact sequences (the horizontal arrows to the right) in the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \xi_{0,b} & \underset{F_b}{\overset{i}{\rightleftarrows}} & \xi_b & \underset{S_b}{\overset{j}{\rightleftarrows}} & \xi_{1,b} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle T_{0,g}} & & \downarrow{\scriptstyle T_g} & & \downarrow{\scriptstyle T_{1,g}} & & \\ 0 & \longrightarrow & \xi_{0,t_g(b)} & \underset{F_{t_g(b)}}{\overset{i}{\rightleftarrows}} & \xi_{t_g(b)} & \underset{S_{t_g(b)}}{\overset{j}{\rightleftarrows}} & \xi_{1,t_g(b)} & \longrightarrow & 0 \end{array} \tag{3.27}$$

If we choose a splitting (F_b, S_b) of the first short exact sequence, then thanks to (3.19) we obtain that the isomorphisms $T_{i,g} : \xi_{i,b} \to \xi_{i,t_g(b)}$ determine a splitting $(F_{t_g(b)}, S_{t_g(b)})$ of the second short exact sequence that satisfies

$$T_{0,g} \circ F_b = F_{t_g(b)} \circ T_g \ , \qquad T_g \circ S_b = S_{t_g(b)} \circ T_{1,g} \ . \tag{3.28}$$

Let $[c] \in H^n(G_b, \xi_{1,b})$, $g \in G$, $g_1 \in G_{t_g(b)}$, …, $g_{n+1} \in G_{t_g(b)}$. Using (3.13), (3.8), (3.17), and (3.28), we obtain

$$\begin{aligned} &\big((\mathrm{Inn}_g, T_{0,g})_{n+1} \circ k_n(c)\big)(g_1, \dots, g_{n+1}) \\ &\qquad = T_{0,g} \circ k_n(c)(\mathrm{Inn}_{g^{-1}}(g_1), \dots, \mathrm{Inn}_{g^{-1}}(g_{n+1})) \\ &\qquad = T_{0,g} \circ F_b \circ T_{\mathrm{Inn}_{g^{-1}}(g_1)} \circ S_b\big(c(\mathrm{Inn}_{g^{-1}}(g_2), \dots, \mathrm{Inn}_{g^{-1}}(g_{n+1}))\big) \\ &\qquad = F_{t_g(b)} \circ T_g \circ T_{g^{-1}} \circ T_{g_1} \circ T_g \circ S_b\big(c(\mathrm{Inn}_{g^{-1}}(g_2), \dots, \mathrm{Inn}_{g^{-1}}(g_{n+1}))\big) \\ &\qquad = F_{t_g(b)} \circ T_{g_1} \circ T_g \circ S_b\big(c(\mathrm{Inn}_{g^{-1}}(g_2), \dots, \mathrm{Inn}_{g^{-1}}(g_{n+1}))\big) \\ &\qquad = F_{t_g(b)} \circ T_{g_1} \circ S_{t_g(b)} \circ T_{1,g}\, c(\mathrm{Inn}_{g^{-1}}(g_2), \dots, \mathrm{Inn}_{g^{-1}}(g_{n+1})) \\ &\qquad = \big(k_n \circ (\mathrm{Inn}_g, T_{1,g})_n(c)\big)(g_1, \dots, g_{n+1}) \ . \end{aligned}$$

This completes the proof that the following part of (3.26) is commutative:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{j_n} & H^n(G, \xi_1)_{\beta,\alpha} & \xrightarrow{k_n} & H^{n+1}(G, \xi_0)_{\beta,\alpha} & \xrightarrow{i_{n+1}} & \cdots \\ & & \downarrow{\scriptstyle \phi^n_{1,\,\alpha\beta}} & & \downarrow{\scriptstyle \phi^{n+1}_{0,\,\alpha\beta}} & & \\ \cdots & \xrightarrow{j_n} & H^n(G, \xi_1)_{\alpha,\beta} & \xrightarrow{k_n} & H^{n+1}(G, \xi_0)_{\alpha,\beta} & \xrightarrow{i_{n+1}} & \cdots \end{array} \tag{3.29}$$

The commutativity of the rest of the diagram (3.26) can be proved analogously.

The commutativity of the diagram (3.26) implies that the transition isomorphisms $\phi^n_{i,\,\alpha\beta}$ determine the long exact sequence

$$0 \longrightarrow \xi_0^G \xrightarrow{i_0} \xi^G \xrightarrow{j_0} \xi_1^G \xrightarrow{k_0} H^1(G, \xi_0) \xrightarrow{i_1} H^1(G, \xi) \xrightarrow{j_1} \cdots \tag{3.30}$$

This answers the question posed in the beginning of this section. Namely, if $k_0(\xi_1^G) = 0$, then the reduced bundles $\xi_i^G = H^0(G, \xi_i)$ form a short exact sequence

$$0 \longrightarrow \xi_0^G \xrightarrow{i_0} \xi^G \xrightarrow{j_0} \xi_1^G \longrightarrow 0 \ , \tag{3.31}$$

hence in this case $\xi^G \cong \xi_0^G \oplus \xi_1^G$. This holds when all finite-dimensional representations of the stationary group G_b are completely reducible for all $b \in B$, or, in particular, when G is compact, because in these cases $H^1(G, \xi_0) = 0$ (i.e., $H^1(G, \xi_0)$ is a bundle with fiber a 0-dimensional vector space). This motivates the following definition.

Definition 3.6. The dimensional reduction of the intertwining short exact sequence (3.18) is said to be *canonical* when the reduced bundles constitute the exact sequence (3.31). Otherwise, we call the nonzero element $k_0(\xi_1^G) \in H^1(G, \xi_0)$ the *obstruction* to canonical dimensional reduction of (3.18).

We recapitulate our results from this section in the following

Theorem 3.7. *Let ξ_0, ξ and ξ_1 be reducible G-vector bundles with the same base B and the same action t of G on B, and let these bundles form an intertwining short exact sequence*

$$0 \longrightarrow \xi_0 \xrightarrow{\ i\ } \xi \xrightarrow{\ j\ } \xi_1 \longrightarrow 0 \ .$$

Then the reduced bundles ξ_0^G, ξ^G and ξ_1^G are a part of the long exact sequence (3.30). *If the obstruction $k_0(\xi_1^G) \in H^1(G, \xi_0)$ to canonical dimensional reduction is zero, then the reduced bundles constitute the short exact sequence* (3.31), *hence in this case*

$$\xi^G \cong \xi_0^G \oplus \xi_1^G \ .$$

Example. Here is an example of a non-canonical reduction of an intertwining short exact of vector bundles. Let the common base B of the vector bundles ξ_0, ξ, and ξ_1 consist of only one point. Let the typical fiber in these bundles be 2-, 3-, and 1-dimensional, respectively. Let G be the Heisenberg group, with elements

$$g_{a,b,c} = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} , \qquad a, b, c \in \mathbb{R} \ ,$$

and the actions on ξ, ξ_0, and ξ_1 be

$$T_{g_{a,b,c}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + ay + bz \\ y + cz \\ z \end{pmatrix} ,$$

$$T_{0,\, g_{a,b,c}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ay \\ y \end{pmatrix} , \qquad T_{1,\, g_{a,b,c}}(z) = (z) \ .$$

Let $i : \xi_0 \to \xi$ be the canonical embedding and $j : \xi \to \xi_1$ be the canonical projection:

$$i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} , \qquad j \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (z) \ .$$

With these definitions, the short exact sequence (3.18) is intertwining with respect to the actions of G.

Let the maps $F : \xi \to \xi_0$ and $S : \xi_1 \to \xi$ be defined by

$$S(z) = \begin{pmatrix} \alpha \\ \beta \\ 1 \end{pmatrix} (z) = \begin{pmatrix} \alpha z \\ \beta z \\ z \end{pmatrix} ,$$

$$F \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - \alpha z \\ y - \beta z \end{pmatrix} ,$$

where α and β are some real constants. Then the pair (F, S) defines a splitting of the short exact sequence (3.18), i.e., the maps F and S satisfy $F \circ i = \mathrm{Id}_{L_0}$ and $\operatorname{im} S = \ker F$.

It is easy to check that the stationary subbundles are

$$\operatorname{st} \xi_0 = \begin{pmatrix} \mathbb{R} \\ 0 \end{pmatrix} , \qquad \operatorname{st} \xi = \begin{pmatrix} \mathbb{R} \\ 0 \\ 0 \end{pmatrix} , \qquad \operatorname{st} \xi_1 = \xi_1 ,$$

and that the G-invariant sections of the three bundles have the form

$$C^\infty(\xi_0)^G = \begin{pmatrix} x \\ 0 \end{pmatrix} , \qquad C^\infty(\xi)^G = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} , \qquad C^\infty(\xi_0)^G = (z) ,$$

where x and z are arbitrary real numbers. Just from counting dimensions, we see that $\xi \not\cong \xi_0 \oplus \xi_1$, i.e., the dimensional reduction of the short exact sequence (3.18) is non-canonical.

Let us consider the long exact sequence (3.30) (cf. (3.16)), and compute the obstruction $k_0(\xi_1^G) \in H^1(G, \xi_0)$ to canonical reduction. Clearly, the injectivity of the canonical embedding i, the fact that both $\operatorname{st} \xi_0$ and $\operatorname{st} \xi$ are one-dimensional, and the exactness of (3.30) imply that $j(\operatorname{st} \xi)$ is zero. By the exactness of (3.30), $k_0 : \operatorname{st} \xi_1 \to H^1(G, \xi_0)$ is injective, therefore the obstruction $k_0(\operatorname{st} \xi_1)$ is one-dimensional.

To compute the obstruction to canonical reduction more explicitly, recall that in this example $H^0(G, \xi_1) = \operatorname{st} \xi_1 = \xi_1$, hence the equivalence class, $[z] \in H^0(G, \xi_1)$, of an element $z \in \operatorname{st} \xi_1 = \xi_1$, is the element z itself. Then $k_0([z])$ is a map from G to ξ_0 defined modulo maps from $\operatorname{im} \delta^{(0)}_{\xi_0}$. According to (3.8), $k_0([z])(g_{a,b,c})$ is the equivalence class of

$$F \circ T_{g_{a,b,c}} \circ S(z) = \begin{pmatrix} (\beta a + b) z \\ c z \end{pmatrix} = z \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\beta \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \beta \\ 1 \end{pmatrix} .$$

The space $\operatorname{im} \delta^{(0)}_{\xi_0}$ consists of maps from G to ξ_0 proportional to

$$A(g_{a,b,c}) = \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \beta \\ 1 \end{pmatrix} .$$

Therefore, we can write $[k_0([z])]\,(g_{a,b,c}) := [k_0([z])(g_{a,b,c})] \in H^1(G,\xi_0)$ as

$$\begin{aligned} [k_0([z])]\,(g_{a,b,c}) &= \begin{pmatrix} (\beta a + b)z \\ cz \end{pmatrix} + \mathbb{R} \begin{pmatrix} a \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} z+\mathbb{R} & 0 & 0 \\ 0 & z & -\beta z \end{pmatrix} g_{a,b,c} \begin{pmatrix} 0 \\ \beta \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbb{R} \\ cz \end{pmatrix} , \end{aligned}$$

which makes explicit the fact that the obstruction to canonical reduction is $k_0(\mathrm{st}\,\xi_1) \cong \mathbb{R}$.

4. Dimensional reduction of tangent and cotangent bundles and their tensor products

4.1. Reduction of $\tau(B)$ and $\tau^*(B)$

4.1.1. Vertical subbundle and reduction of $\tau(B)$. Let B be a manifold endowed with an action t of the Lie group G which satisfies Conditions A and B from Section 2.2. Let $\tau(B) = (T(B),\pi,B)$ and $\tau^*(B)$ be respectively the tangent and the cotangent bundles to B. Let $t_* : G \times T(B) \to T(B)$ be the tangent lift of the action t. Then (t_*, t) is an action of G on $\tau(B)$ through vector bundle morphisms. This action makes $\tau(B)$ a reducible G-vector bundle.

The base B becomes the total space of the G-bundle (B,p,N) as in (2.11). Let $\tau^{\mathrm{v}}(B) \subseteq \tau(B)$ be the *vertical subbundle* of $\tau(B)$, which by definition consists of the vectors tangent to the fibers of (B,p,N):

$$\tau^{\mathrm{v}}(B)_b := \tau(p^{-1}(p(b)))_b \qquad \forall\, b \in B\ .$$

Let $p^*\tau(N)$ be the bundle over N induced by the projection $p : B \to N$. If we think of $p^*\tau(N)$ as the set of all pairs $(b,w) \in B \times T(N)$ satisfying $p(b) = \pi_N(w)$ (where $\pi_N : T(N) \to N$ is the natural projection), then the action of $g \in G$ is given by

$$(b,w) \mapsto (t_g(b), w)\ . \tag{4.1}$$

Let $i : \tau^{\mathrm{v}}(B) \to \tau(B)$ be the natural embedding, and $j : \tau(B) \to p^*\tau(N)$ be the natural projection. Then the short exact sequence

$$0 \longrightarrow \tau^{\mathrm{v}}(B) \xrightarrow{\ i\ } \tau(B) \xrightarrow{\ j\ } p^*\tau(N) \longrightarrow 0 \tag{4.2}$$

is intertwining with respect to the corresponding actions of G. Hence, to perform G-dimensional reduction of (4.2), we can apply Theorem 3.7, and then the only remaining task is to find the obstruction $k_0(p^*\tau(N)) \in H^1(G,\tau^{\mathrm{v}}(B))$ for canonical dimensional reduction.

The obstruction for canonical reduction will be zero if for each $b \in B$ the G_b-invariant subspace $\tau^{\mathrm{v}}(B)_b$ has a G_b-invariant complement in $\tau(B)_b$. For the particular case of the tangent lift t_* of the action t of G on B, the question of finding this obstruction is easy. Indeed, Condition A guarantees that (B,π,N) (see (2.11)) is a locally trivial G-bundle. Because of this, there exists a local section σ of (B,π,N) whose graph $\widetilde{U}$ contains b, and such that the stationary group $G_{b'}$ of each point $b' \in \widetilde{U}$ is the same as the stationary group G_b of b. More explicitly, if $V \subseteq N$ is an open subset of N containing $p(b)$, then

$$W^{(b)} := \{ \, \Phi^{-1}(x, \pi_2 \circ \Phi(b)) \, : \, x \in V \subseteq N \, \}$$

is a submanifold of B which is transversal to the orbits of G in B, contains b, and consists of points whose stationary group is G_b (we used the notation of (2.13)). This implies that G_b acts trivially on $W^{(b)}$, which in turn means that G_b acts trivially on $\tau(W^{(b)})$, thus $\tau(W^{(b)})_b$ is a G-invariant subspace of $\tau(B)_b$ complementary to $i(\tau^{\mathrm{v}}(B)_b)$. Therefore the obstruction $k_0(p^*\tau(N))$ is zero, and Theorem 3.7 yields the short exact sequence

$$0 \longrightarrow \mathrm{st}\,\tau^{\mathrm{v}}(B)_b \xrightarrow{\ i\ } \mathrm{st}\,\tau(B)_b \xrightarrow{\ j\ } p^*\tau(N)_b \longrightarrow 0 \ .$$

Finally we notice that $p^*\tau(N)^G \cong \tau(N)$, and summarize the above in the following

Lemma 4.1. *Let the action t of the group G on the manifold B satisfy Condition A from Section* 2.2. *Then the intertwining with respect to the tangent lift (t_*, t) short exact sequence* (4.2) *of reducible G-vector bundles has canonical dimensional reduction*

$$0 \longrightarrow \tau^{\mathrm{v}}(B)^G \xrightarrow{\ i\ } \tau(B)^G \xrightarrow{\ j\ } \tau(N) \longrightarrow 0 \ . \qquad (4.3)$$

4.1.2. G-invariant connections and reduction of $\tau(B)$. The problem of splitting the short exact sequence (4.3) is directly related to the problem of description of all G-invariant connections. A connection on the bundle (B, p, N) can be defined by specifying a "horizontal" subbundle $\tau^{\mathrm{h}}(B) \subseteq \tau(B)$ such that

$$\tau(B) = \tau^{\mathrm{v}}(B) \oplus \tau^{\mathrm{h}}(B) \ . \qquad (4.4)$$

The connection is *G-invariant* if $(t_g)_*(\tau^{\mathrm{h}}(B)_b) = \tau^{\mathrm{h}}(B)_{t_g(b)}$ for each $g \in G$ and $b \in B$. The parallel transport of a G-invariant connection preserves the structure of a homogeneous G-space in the fibers of (B, p, N). Hence, the G-invariant connections on (B, p, N) correspond to the connections on the principal bundle $P(\mathcal{N}(H)/H, N)$ with which (B, p, N) is associated (recall the discussion after formulating Condition A).

Let (4.4) define a G-invariant connection on (B, p, N). For each point $b \in B$, the horizontal subspace $\tau^{\mathrm{h}}(B)_b$ consists of vectors that are stationary with respect of the representations of G_b in $\tau(B)_b$. Indeed, let γ be a path in N through the point $p(b)$, and $\widetilde{\gamma}$ be its horizontal lift in B passing through b. Because of the G-invariance of the connection, the stationary group $G_{b'}$ of any point $b' \in \widetilde{\gamma}$ will be the same as G_b, i.e., $\widetilde{\gamma}$ is invariant with respect to the action of G_b, and $\tau(\widetilde{\gamma})_b \subseteq \tau^{\mathrm{h}}(B)_b$ consists of stationary vectors. Clearly, all vectors in $\tau^{\mathrm{h}}(B)_b$ are tangent to the horizontal lift of some path through $p(b)$, hence $\tau^{\mathrm{h}}(B) \subseteq \mathrm{st}\,\tau(B)$.

Consider the isomorphism

$$\tau^{\mathrm{h}}(B) \to p^*\tau(N) : u \mapsto (\pi(u), p_* u) \ , \qquad (4.5)$$

where $u \in \tau^h(B)$, $\pi : \tau(B) \to B$ is the natural projection in $\tau(B)$, $p_* : \tau(B) \to \tau(N)$ is the tangent lift of $p : B \to N$, and we think of the pair $(\pi(u), p_* u)$ as an element of $p^*\tau(N)$ (because it satisfies $p(\pi(u)) = \pi_N(p_* u)$). Thanks to (4.1) and the fact that $\tau^{\mathrm{h}}(B) \subseteq \mathrm{st}\,\tau(B)$, the map (4.5) is an isomorphism of G-bundles.

The G-invariant connection (4.4) endows the coordinate realizations of $\tau(B)^G$ with a structure of a direct sum:

$$\mathrm{st}\,\tau(B)\restriction_{\widetilde{U}_\alpha} = \mathrm{st}\,\tau^{\mathrm{v}}(B)\restriction_{\widetilde{U}_\alpha} \oplus \tau^{\mathrm{h}}(B)\restriction_{\widetilde{U}_\alpha} \ .$$

The transition isomorphisms $\phi_{\alpha\beta}$ (2.23) preserve this direct sum and yield a direct sum of the reduced bundles

$$\tau(B)^G = \tau^{\mathrm{v}}(B)^G \oplus \tau^{\mathrm{h}}(B)^G = \tau^{\mathrm{v}}(B)^G \oplus \tau(N) \ ,$$

i.e., a splitting of the short exact sequence (4.3).

Conversely, suppose that we are given a splitting of the short exact sequence (4.3). Then each coordinate realization of $\tau(B)^G$ becomes a direct sum

$$\mathrm{st}\,\tau(B)\restriction_{\widetilde{U}_\alpha} = \mathrm{st}\,\tau^{\mathrm{v}}(B)\restriction_{\widetilde{U}_\alpha} \oplus \zeta_\alpha \ ,$$

where ζ_α are subbundles of $\tau(B)$ over $\widetilde{U}_\alpha$ such that $\dim \zeta_\alpha = \dim N$, $\zeta_\alpha \subseteq \mathrm{st}\,\tau(B)\restriction_{\widetilde{U}_\alpha}$, ζ_α are complementary to $\tau^{\mathrm{v}}(B)_{\widetilde{U}_\alpha}$, and satisfy $\phi_{\alpha\beta}(\zeta_\beta) = \zeta_\alpha$. Then there exists a unique G-invariant connection such that $\tau^{\mathrm{h}}(B)\restriction_{\widetilde{U}_\alpha} = \zeta_\alpha$. It is clear that different G-invariant connections correspond to different splittings of (4.3), and vice versa. Thus, we obtain the following

Theorem 4.2. *There is a bijective correspondence between all G-invariant connections on (B, p, N) (or, equivalently, all connections on the principal bundle $P(\mathcal{N}(H)/H, N)$) and all splittings of the short exact sequence* (4.3).

Because of Theorem 4.2, the set of all G-invariant connections on (B, p, N) is an affine space with linear group $\mathrm{Hom}\big(\tau(N), \tau^{\mathrm{v}}(B)^G\big) = C^\infty\big(\tau^*(N) \otimes \tau^{\mathrm{v}}(B)^G\big)$.

4.1.3. Reduction of $\tau^*(B)$. The case of a cotangent bundle is similar to the reduction of $\tau(B)$, but some aspects are different. The action (t_*, t) of G on $\tau(B)$ naturally determines the contragradient action $(\hat{t}, t) := (t^{*\,-1}, t)$ of G on $\tau^*(B)$. The complete reducibility of (t_*, t) implies the complete reducibility of $(\hat{t}, t)$, which yields

Lemma 4.3. *If the assumptions of Lemma* 4.1 *are satisfied, then the intertwining short exact sequence*

$$0 \longleftarrow \tau^{\mathrm{v}}(B)^* \xleftarrow{\ i^*\ } \tau^*(B) \xleftarrow{\ j^*\ } (p^*\tau(N))^* \longleftarrow 0 \ , \qquad (4.6)$$

has a canonical dimensional reduction

$$0 \longleftarrow (\tau^{\mathrm{v}}(B)^*)^G \xleftarrow{\ i^*\ } \tau^*(B)^G \xleftarrow{\ j^*\ } \tau^*(N) \longleftarrow 0 \ . \qquad (4.7)$$

In general, it is not true that $(\tau^{\mathrm{v}}(B)^G)^* \cong (\tau^{\mathrm{v}}(B)^*)^G$, and the analogue of Theorem 4.2 does not hold – this is an instance of a "violation of duality" between the tangent and cotangent bundles in the process of reduction, mentioned at the end of the Introduction; we give an example of this phenomenon in Section 4.1.5. Let us compare $(\xi^G)^*$ and $(\xi^*)^G$ in the general case of a reducible G-vector bundle ξ and its dual ξ^*. By definition, the fiber of ξ^* over any point $b \in B$ is the dual space of ξ_b, and the action $(\hat{T}, t)$ of G on ξ^* is the contragradient action to the action (T, t) of G on ξ. In general the dimensions of $\mathrm{st}\,\xi_b$ and $(\mathrm{st}\,\xi^*)_b$ are not equal. Assume that G has only completely reducible finite-dimensional representations. Then $\mathrm{st}\,\xi_b \subseteq \xi_b$ has a uniquely defined G_b-invariant complement L_b, hence the canonical projections $\pi_1 : \xi_b \to \mathrm{st}\,\xi_b$ and $\pi_2 : \xi_b \to L_b$ are well-defined. In the dual space, $\pi_1^* : (\mathrm{st}\,\xi_b)^* \to \xi_b^*$ and $\pi_2^* : L_b^* \to \xi_b^*$ define a direct sum $\xi_b^* = \pi_1^*(\mathrm{st}\,\xi_b)^* \oplus \pi_2^*(\xi_b^*)$. The image $\pi_1^*((\mathrm{st}\,\xi_b)^*)$ coincides with $(\mathrm{st}\,\xi^*)_b$ and defines a natural isomorphism $\pi_1^* : (\mathrm{st}\,\xi_b)^* \to (\mathrm{st}\,\xi^*)_b$, hence $(\xi^G)^* = (\xi^*)^G$. Therefore, if the action t of G on B is such that the stationary group G_b has only completely reducible finite-dimensional representations for any $b \in B$ (which is the case when G_b is compact), then $(\tau^{\mathrm{v}}(B)^G)^* = (\tau^{\mathrm{v}}(B)^*)^G$ will hold. Simple reasoning shows that $\tau^*(N) = (p^*\tau(N)^G)^* = ((p^*\tau(N)^*)^G$ (which was already taken into consideration in (4.7)). Therefore we obtain the following theorem.

Theorem 4.4. *Let the action t of G on B satisfy the assumptions of Lemma* 4.1*, and let the stationary group G_b have only completely reducible finite-dimensional representations. Then $(\tau^{\mathrm{v}}(B)^*)^G = (\tau^{\mathrm{v}}(B)^G)^*$, and there exists a natural bijective correspondence between all G-invariant connections on* (B, p, N) *(or, equivalently, all connections on the principal bundle* $P(\mathcal{N}(H)/H, N)$*) and all splittings of the short exact sequence*

$$0 \longleftarrow (\tau^{\mathrm{v}}(B)^G)^* \xleftarrow{i^*} \tau^*(B)^G \xleftarrow{j^*} \tau^*(N) \longleftarrow 0\,.$$

Under the conditions of Theorem 4.4, the set of all G-invariant connections on (B, p, N) is an affine space with linear group $\mathrm{Hom}\big((\tau^{\mathrm{v}}(B)^G)^*, \tau^*(N)\big) = C^\infty\big(\tau^{\mathrm{v}}(B)^G \otimes \tau^*(N)\big)$.

4.1.4. Short exact sequences related to G-invariance of submanifolds. Here we state a simple but important fact which helps identify invariant subbundles. Let C be a submanifold of the manifold B that is invariant with respect to the action t of G on B, i.e., $t_g(C) \subseteq C$ for each $g \in G$. Then the restriction of the base to C yields naturally the short exact sequence

$$0 \longrightarrow \tau(C) \xrightarrow{m} \tau(B)_C \xrightarrow{n} \nu(C) \longrightarrow 0 \qquad (4.8)$$

of vector bundles over C. Here $m : C \hookrightarrow B$ is the natural embedding, $\nu(C) := \tau(B)_C/\tau(C)$, and n is the natural projection. The short exact sequence (4.8) and its dual are G-intertwining, which implies that the subbundles $m(\tau(C))$ and $n^*(\nu^*(C))$ are G-invariant subbundles of $\tau(B)_C$ and $\tau^*(B)_C$, respectively.

4.1.5. An example of "violation of duality" in the reduction of $\tau(\xi)^G$ and $\tau^*(\xi)^G$. In the Appendix of our paper [3], we construct an explicit example of an intertwining short exact sequence of the form (4.2) for which $(\tau^{\mathrm{v}}(B)^G)^* \ncong (\tau^{\mathrm{v}}(B)^*)^G$ – a possibility we discussed in Section 4.1.3. Here we briefly describe it, referring the reader [3] for detailed computations.

Let $\mathbb{R}^6_{2,4}$ be the real 6-dimensional space with diagonal metric tensor with two entries equal to (-1) and four entries equal to 1, and let $G = \mathrm{O}_0(2,4)$ be the connected component of the orthogonal group in $\mathbb{R}^6_{2,4}$. The future light cone B in $\mathbb{R}^6_{2,4}$ (denoted by Q^6 in [3]) is a G-invariant submanifold of $\mathbb{R}^6_{2,4}$. Consider the corresponding short exact sequences (4.2) and its dual, (4.6), both of which are intertwining with respect to the corresponding actions of G. Since the action of G on B is transitive, the base N of the reduced bundles consists of a single point, for a realization of which we take a particular point $y \in B$. The action of G on B is also free, so $\tau^{\mathrm{v}}(B) = \tau(B)$ and, hence, $\tau^{\mathrm{v}}(B)^* = \tau^*(B)$. In [3] we compute the stationary group G_y, and show that G_y is non-compact (which is crucial for this example to work). The dimensions of (the fibers of) the stationary subbundles turn out to be $\dim \mathrm{st}\, \tau(B)_y = 1$ and $\dim \mathrm{st}\, \tau^*(B)_y = 0$. Therefore the dimensions of (the fibers of) the reduced vertical subbundles $\tau^{\mathrm{v}}(B)^G = \tau(B)^G$ and $(\tau^{\mathrm{v}}(B)^*)^G = \tau^*(B)^G$ are 1 and 0, respectively, so they cannot be dual to one another.

4.2. Dimensional reduction of tensor products of $\tau(B)$ and $\tau^*(B)$

Let $\otimes^k\xi$, $S^k\xi$, $\Lambda^k\xi$, and $\Gamma\xi$ stand for the kth tensor product of ξ, kth symmetric or antisymmetric tensor product of ξ, and the tensor product of ξ with symmetry determined by some Young tableau. Below we consider their dimensional reduction, assuming, as above, that the actions of G on $\tau(B)$ and $\tau^*(B)$ are respectively the tangent and cotangent lifts of the action of G on the base B.

When the action t of g of G on B is free, computing $(\otimes^k\tau(B))^G$, $(S^k\tau(B))^G$, ... is a purely algebraic procedure thanks to Lemmata 2.11 and 2.12. Here we treat in detail the reduction of $(S^k\tau(B))^G$ in this case. Recall that base of the reduced bundle is glued from the local realizations $\widetilde{U}_\alpha \subseteq B$ (2.17), and the fibers of the reduced bundle are the stationary subspaces (cf. (2.18)). If $b \in \widetilde{U}_\alpha$, then the fact that the action of G on B is free implies that $(\xi^G)_b = \mathrm{st}\, \xi_b = \xi_b$. Therefore, $((S^k\tau(B))^G)_b = \mathrm{st}\,(S^k\tau(B))_b = S^k(\mathrm{st}\, \tau(B)_b) = S^k((\tau(B)^G)_b)$, which can be written shortly as $(S^k\tau(B))^G = S^k(\tau(B)^G)$. Using this observation and Lemma 4.1, we obtain

$$(S^k\tau(B))^G = S^k(\tau(B)^G) \cong S^k\left[\tau^{\mathrm{v}}(B)^G \oplus \tau(N)\right] = \bigoplus_{i=0}^{k} \left[S^i(\tau^{\mathrm{v}}(B)^G) \otimes S^{k-i}\tau(N)\right], \tag{4.9}$$

i.e., the reduced bundle $(S^k\tau(B))^G$ is isomorphic to a Whitney sum of tensor products of vector bundles of type $S^i(\tau^{\mathrm{v}}(B)^G) \otimes S^{k-i}\tau(N)$. The isomorphism in (4.9) is not natural, i.e., it depends on the choice of a splitting of (4.3) (or, equivalently, on the choice of a G-invariant connection of (B, p, N) (2.11)).

When the action t of G on B is not free, we have to use Lemmata 2.11 and 2.13. In this case, the reduction of $S^k\tau(B)$ is

$$(S^k\tau(B))^G \cong \bigoplus_{i=0}^{k}\Big[\big(S^i\tau^{\mathrm{v}}(B)\big)^G \otimes S^{k-i}\tau(N)\Big] . \tag{4.10}$$

It is important to notice that, if the action t is not free, it may happen that $(S^i\tau^{\mathrm{v}}(B))^G \neq S^i(\tau^{\mathrm{v}}(B)^G)$. Here are some details of the derivation of (4.10). A splitting of (4.3) yields a representation (4.4) of $\tau(B)$ as a G-equivariant direct sum $\tau(B) = \tau^{\mathrm{v}}(B) \oplus \tau^{\mathrm{h}}(B)$ which, in turn, gives

$$S^k\tau(B) \cong \bigoplus_{i=0}^{k}\Big[S^i\tau^{\mathrm{v}}(B) \otimes S^{k-i}\tau^{\mathrm{h}}(B)\Big] ,$$

where each term in the direct sum in the right-hand side is a G-invariant subbundle of $S^k\tau(B)$. Taking into account that, for all $j = 0, 1, \ldots, k$, the stationary group G_b of $b \in B$ acts trivially on $S^j\tau^{\mathrm{h}}(B)_b$ – and, therefore, st $S^j\tau^{\mathrm{h}}(B)_b = S^j\tau^{\mathrm{h}}(B)_b$ – we apply Lemmata 2.11 and 2.13 to obtain (4.10).

If we do not use a splitting of (4.3), then the only structures that occur naturally in the reduction of the tensor products of $\tau(B)$ come from the embedding $\tau^{\mathrm{v}}(B)^G \hookrightarrow \tau(B)^G$. For example, the only natural structure in $(S^k\tau(B))^G$ is the sequence of embeddings

$$(S^k\tau^{\mathrm{v}}(B))^G \hookrightarrow (S^{k-1}\tau^{\mathrm{v}}(B))^G \otimes \tau(B)^G \hookrightarrow (S^{k-2}\tau^{\mathrm{v}}(B))^G \otimes (S^2\tau(B))^G \hookrightarrow \cdots$$
$$\cdots \hookrightarrow \tau^{\mathrm{v}}(B)^G \otimes (S^{k-1}\tau(B))^G \hookrightarrow (S^k\tau(B))^G .$$

The cases of $(\otimes^k\tau(B))^G$, $(\Lambda^k\tau(B))^G$, and $(\Gamma^k\tau(B))^G$ are completely analogous.

For the other basic cases we obtain

$$(\Lambda^k\tau(B))^G \cong \bigoplus_{i=0}^{k}\Big[\big(\Lambda^i\tau^{\mathrm{v}}(B)\big)^G \otimes \Lambda^{k-i}\tau(N)\Big] ,$$

$$(S^k\tau^*(B))^G \cong \bigoplus_{i=0}^{k}\Big[\big(S^i\tau^{\mathrm{v}}(B)^*\big)^G \otimes S^{k-i}\tau^*(N)\Big] , \tag{4.11}$$

$$(\Lambda^k\tau^*(B))^G \cong \bigoplus_{i=0}^{k}\Big[\big(\Lambda^i\tau^{\mathrm{v}}(B)^*\big)^G \otimes \Lambda^{k-i}\tau^*(N)\Big] . \tag{4.12}$$

Similarly to the case of the tangent bundle, if a splitting of the short exact sequence (4.7) is not chosen, then the only natural structures occurring in the reduction of the tensor products of $\tau^*(B)$ come from the embedding $\tau^*(N) \hookrightarrow \tau^*(B)^G$. The only natural structure in $(S^k\tau^*(B))^G$ is the sequence of embeddings

$$S^k\tau^*(N) \hookrightarrow S^{k-1}\tau^*(N) \otimes \tau^*(B)^G \hookrightarrow S^{k-2}\tau^*(N) \otimes (S^2\tau^*(B))^G \hookrightarrow \cdots$$
$$\cdots \hookrightarrow \tau^*(N) \otimes (S^{k-1}\tau^*(B))^G \hookrightarrow (S^k\tau^*(B))^G ;$$

the cases of $(\otimes^k\tau^*(B))^G$, $(\Lambda^k\tau^*(B))^G$, and $(\Gamma^k\tau^*(B))^G$ are analogous.

An essential difference between the cases of tangent and cotangent bundles is that, in general, the G-invariant connections on (B, p, N) is not parametrized by the set of all splittings of (4.7), $\mathrm{Hom}\big((\tau^{\mathrm{v}}(B)^*)^G, \tau^*(N)\big)$. This is another manifestation of the breaking the duality between the tangent and cotangent bundles in the process of dimensional reduction. If, however, the action t of G on B is such that G_b has only completely reducible finite-dimensional representations, then the G-invariant connections on (B, p, N) are indeed parametrized by $\mathrm{Hom}\big((\tau^{\mathrm{v}}(B)^*)^G, \tau^*(N)\big)$ according to Theorem 4.4.

4.3. Dimensional reduction of invariant tensor fields

The results in Sections 4.1 and 4.2 can be written in terms of sections of the corresponding vector bundles.

Consider first the case of the tangent and cotangent bundles. The local coordinates x^μ, $\mu = 1, 2, \ldots, \dim B$ in B generate local coordinates $(x^\mu, \mathrm{d}x^\mu)$ in $T(B)$ and $(x^\mu, \frac{\partial}{\partial x^\mu})$ in $T^*(B)$. In these coordinates a vector field $X \in C^\infty(\tau(B))$ and a one-form $A \in C^\infty(\tau^*(B))$ are G-invariant if for each $g \in G$ and $b \in B$

$$g(X)^\mu(b) \equiv \frac{\partial t_g^\mu(t_{g^{-1}}(b))}{\partial x^\nu} X^\nu(t_{g^{-1}}(b)) \;=\; X^\mu(b) \ ,$$

$$g(A)_\mu(b) \equiv \frac{\partial t^\nu_{g^{-1}}(b)}{\partial x^\mu} A_\nu(t_{g^{-1}}(b)) \;=\; A_\mu(b) \ .$$

According the the general construction (Lemma 4.1), the set of G-invariant vector fields is in a bijective correspondence with the sections of the reduced bundle, $\tau(B)^G \cong \tau^{\mathrm{v}}(B)^G \oplus \tau(N)$. In other words, there exists a bijective correspondence between the G-invariant vector fields and the pairs of a "scalar field" (i.e., a section of $\tau^{\mathrm{v}}(B)^G$) and a vector field on N. This isomorphism is not natural in the sense that the original setup does not determine a splitting of (4.3). If, however, the whole construction is a part of a physics problem, the physical interpretation may provide us with additional information which may possibly single out some splitting of (4.3). If this is the case, then the G-invariant vector fields on B correspond to the sections of $\tau(B)^G$ in which there exists an additional structure, namely, some "scalar" subbundle $\tau^{\mathrm{v}}(B)^G \subseteq \tau(B)^G$ with the property $\tau(B)^G/\tau^{\mathrm{v}}(B)^G \cong \tau(N)$.

The interpretation of the isomorphisms and embeddings from Section 4.2 is analogous. For example, (4.12) can be interpreted as giving a bijective correspondence between the G-invariant k-forms on B and the following set of $(k+1)$ terms:

- a "scalar" field, i.e., a section of $(\Lambda^k \tau^{\mathrm{v}}(B)^*)^G$,
- a 1-form on N with coefficients in the vector bundle $(\Lambda^{k-1}\tau^{\mathrm{v}}(B)^*)^G$, i.e., a section of $(\Lambda^{k-1}\tau^{\mathrm{v}}(B)^*)^G \otimes \tau^*(N)$,
- a 2-form on N with coefficients in the vector bundle $(\Lambda^{k-2}\tau^{\mathrm{v}}(B)^*)^G$, i.e., a section of $(\Lambda^{k-2}\tau^{\mathrm{v}}(B)^*)^G \otimes \Lambda^2\tau^*(N)$,
- $\ldots$,
- a $(k-1)$-form on N with coefficients in $(\tau^{\mathrm{v}}(B)^*)^G$, and

- a k-form on N.

An important particular case is the Marsden-Weinstein-Meyer reduction of symmetric Hamiltonian systems [14, 15]. In this case $(\widetilde{B}, \widetilde{\omega})$ is a symplectic manifold, and a Lie group $\widetilde{G}$ acts by symplectomorphisms (i.e., $t^*_{\widetilde{g}}\widetilde{\omega} = \widetilde{\omega}$ for all $\widetilde{g} \in \widetilde{G}$). Let $\widetilde{\Phi} : \widetilde{B} \to \widetilde{\mathfrak{g}}^*$ be the moment map (where $\widetilde{\mathfrak{g}}$ is the Lie algebra of $\widetilde{G}$), $\widetilde{a} \in \widetilde{\mathfrak{g}}^*$ be a regular value of $\widetilde{\Phi}$, and $G := \{\widetilde{g} \in \widetilde{G} : \mathrm{Ad}^*_{\widetilde{g}}(\widetilde{a}) = \widetilde{a}\}$ be the stationary group of $\widetilde{G}$ with respect to the coadjoint representation of $\widetilde{G}$ in $\widetilde{\mathfrak{g}}^*$. Then $B := \widetilde{\Phi}^{-1}(\widetilde{a})$ is a submanifold of $\widetilde{B}$, and G acts naturally on B; assume that this action satisfies the conditions from Section 2.2. Clearly, the restriction of $\widetilde{\omega}$ to B is G-invariant and, according to (4.12), corresponds to a triple $(\omega_1, \omega_2, \omega_3)$, where $\omega_1 \in C^\infty((\Lambda^2\tau^{\mathrm{v}}(B)^*)^G)$, $\omega_2 \in C^\infty((\tau^{\mathrm{v}}(B)^*)^G \otimes \tau^*(N))$, $\omega_3 \in C^\infty(\Lambda^2\tau^*(N))$, and $N := B/G$. In this language, the theorem of symplectic reduction claims that ω_1 and ω_2 are zero, and ω_3 is a symplectic form on N.

Another case important in theoretical physics is the dimensional reduction of invariant metrics, and, in particular, Kaluza-Klein-type theories. Kaluza [16] proposed to embed the space-time in a manifold of higher dimension, impose a certain (Abelian) symmetry on the system, and consider objects on the space-time coming from higher-dimensional objects. This idea was pursued by Kerner [17] who generalized it to non-Abelian symmetries; see also the famous Problem 77 in the work of DeWitt [18] (all these and many other references can be found in the collection [19]). Kaluza-Klein techniques have been very popular since late 1970's among relativists and quantum field theorists. In our formalism, the reduction of invariant metrics is governed by the isomorphism

$$(S^2\tau^*(B))^G \cong (S^2\tau^{\mathrm{v}}(B)^*)^G \oplus \left[(\tau^{\mathrm{v}}(B)^*)^G \otimes \tau^*(N)\right] \oplus S^2\tau^*(N)$$

(see (4.11)). This is in agreement with the classical result saying that, under certain non-degeneracy conditions, the G-invariant metrics on B are in a bijective correspondence with the triples of a "scalar field" (a section of $(S^2\tau^{\mathrm{v}}(B)^*)^G$), a linear connection with values in the Lie algebra of G, and a metric on N (a detailed treatment for compact groups G can be found in the papers of of Coquereaux and Jadczyk [20, 21, 22] and in their book [23]). Without going into details, we would like to note that in the non-Euclidean case this correspondence is not bijective in general because the non-degeneracy conditions may be violated.

Acknowledgements

The research of NPP and his visits to the University of Sofia have been partially supported by National Science Foundation grants DMS-0405903 and DMS-0807658.

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