Nonhomogeneous linear ODE of order n with constant coefficients

We are looking for the general solution of the nonhomogeneous equation

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = f(x), \qquad (N)$$

where $a_0, \dots, a_n$ are constants, $a_n \neq 0$.

Let

$$a_n y^{(n)} + \dots + a_1 y' + a_0 y = 0 \qquad (H)$$

be the corresponding homogeneous eqn, and

$$a_n r^n + \dots + a_1 r + a_0 = 0 \qquad (C)$$

be the characteristic eqn of (H).

Notation:

$$L := a_n D^n + \dots + a_1 D + a_0,$$

then (N) and (H) can be written as $Ly = f(x)$ and $Ly = 0$, respectively.

Two important rules:

1) $$\begin{pmatrix} \text{general sol'n} \\ y(x) \text{ of} \\ Ly = f(x) \end{pmatrix} = \begin{pmatrix} \text{general sol'n} \\ y_c(x) \text{ of} \\ Ly = 0 \end{pmatrix} + \begin{pmatrix} \text{particular sol'n} \\ y_p(x) \text{ of} \\ Ly = f(x) \end{pmatrix}$$

2) If (N) reads
$$Ly = f_1(x) + \cdots + f_r(x),$$
then if $y_{p,1}(x), \dots, y_{p,r}(x)$ are solutions of
$$Ly = f_1(x), \dots, Ly = f_r(x),$$
resp., then
$$y_p(x) := y_{p,1}(x) + \cdots + y_{p,r}(x)$$
is a solution of
$$Ly = f_1(x) + \cdots + f_r(x).$$

Therefore, to find the general solution of $Ly = f(x)$, one has to find:

- the general solution of $Ly = 0$ (which is a standard procedure), and
- one particular solution of $Ly = f(x)$ (which is more complicated, and is explained below in some particular cases).

Method of undetermined coefficients for finding a particular sol'n of $Ly = f$

- Let
$$f(x) = e^{cx} P_m(x).$$
Then, if c is a root of (C) with multiplicity s, look for a particular solution of (N) of the form
$$y_p(x) = x^s e^{cx} Q_m(x)$$
(where $Q_m(x)$ is a polynomial of degree m).

- Let
$$f(x) = e^{cx}\left[P_{m_1}(x)\cos dx + P_{m_2}(x)\sin dx\right].$$
Then, if $c+id$ is a root of (C) with multiplicity s, define
$$m := \max(m_1, m_2),$$
and look for a particular solution of (N) of the form
$$y_p(x) = x^s e^{cx}\left[Q_m(x)\cos dx + \tilde{Q}_m(x)\sin dx\right],$$
where Q_m and $\tilde{Q}_m$ are polynomials of degree m.

Example:

$$y'' - 5y' + 6y = 2e^{3x}$$

- $r^2 - 5r + 6 = 0 \longrightarrow$ the roots of the characteristic equation are 2 & 3 (both simple), hence the general sol'n of the associated homogeneous eqn, $y'' - 5y' + 6y = 0$, is

$$y_c(x) = C_1 e^{2x} + C_2 e^{3x}.$$

- $f(x) = 2e^{3x}$ is of the form $e^{cx} P_m(x)$ with $c = 3$, $m = 0$. Since 3 is a root of (C) of multiplicity 1, we look for a particular solution of the nonhom. eqn. of the form

$$y_p(x) = x^1 e^{3x} P_0(x) = A x e^{3x}.$$

To determine the constant A, we plug $y_p(x)$ in the nonhomog. eqn:

$$y_p'(x) = A(1+3x)e^{3x}$$

$$y_p''(x) = 3A(2+3x)e^{3x}$$

(it is a great exercise to reproduce all these calculations), and substitute all these expressions in

$$y_p'' - 5y_p' + 6y_p = 2e^{3x}:$$

$$3A(2+3x)e^{3x} - 5A(1+3x)e^{3x}$$
$$+6Axe^{3x} = 2e^{3x}$$

which, after expanding all expressions and simplifying, becomes

$$A + 0x = 2e^{3x} \Rightarrow A = 2,$$

so the function

$$y_p(x) = 2xe^{3x}$$

is a particular solution of (N).
Therefore, the general solution of (N) is

$$y(x) = C_1 e^{2x} + C_2 e^{3x} + 2xe^{3x}.$$

<u>Example:</u>

$$y''' - y'' + y' - y = (x+1)e^{2x}$$

- $r^3 - r^2 + r - 1 = 0$

  $(r-1)(r^2+1) = 0$

  $\Rightarrow \begin{cases} 1 \text{ is a root of mult. } 1, \\ \pm i \text{ are roots of mult. } 1 \text{ each.} \end{cases}$

  $\Rightarrow y_c(x) = C_1 e^x + C_2 \cos x + C_3 \sin x.$

- $f(x) = (x+1)e^{2x} = e^{2x} P_1(x)$,
  and 2 is <u>not</u> a root of the char. eqn,

so we are looking for $y_p(x)$ of the form

$$y_p(x) = x^0 e^{2x} Q_1(x) = e^{2x}(Ax+B).$$

To determine the values of A and B, we compute:

$$y_p'(x) = e^{2x}(2Ax + 2B + A),$$

$$y_p''(x) = e^{2x}(4Ax + 4B + 4A)$$

$$y_p'''(x) = e^{2x}(8Ax + 8B + 12A),$$

and substitute in the nonhom. eqn:

$$e^{2x}(8Ax + 8B + 12A) - e^{2x}(4Ax + 4B + 4A)$$
$$+ e^{2x}(2Ax + 2B + A) - e^{2x}(Ax + B)$$
$$= (x+1)e^{2x}$$

$$\Rightarrow 5Ax + 9A + 5B = x + 1,$$

equating the coefficients in front of x^1 and x^0, we obtain the system

$$\begin{cases} 5A = 1 \\ 9A + 5B = 1, \end{cases}$$

whose solution is

$$A = \frac{1}{5}, \quad B = -\frac{4}{25}.$$

Therefore, a particular solution of the nonhom. eqn is

$$y_p(x) = e^{2x}\left(\frac{x}{5} - \frac{4}{25}\right),$$

and the general solution of the nonhom. eqn. is

$$y(x) = C_1 e^x + C_2 \cos x + C_3 \sin x + e^{2x}\left(\frac{x}{5} - \frac{4}{25}\right).$$

Example:

$$y^{(3)} + y' = 2 - \sin x$$

- $r^3 + r = 0 \Rightarrow r(r^2+1) = 0$

$\Rightarrow \begin{cases} 0 \text{ is a root of mult. } 1, \\ \pm i \text{ are roots of mult. } 1 \text{ each.} \end{cases}$

$\Rightarrow y_c(x) = C_1 + C_2 \cos x + C_3 \sin x.$

- look for a particular solution, $y_{p,1}(x)$, of

$$y^{(3)} + y' = 2 = e^{0x} P_0(x).$$

Since 0 is a root of the char. eqn. of mult. 1, look for $y_{p,1}(x)$ of the form

$$y_{p,1}(x) = x^1 e^{0x} P_0(x) = Ax.$$

Substituting $y_{p,1}(x)$ in $y^{(3)} + y' = 2$, we obtain $A=2$

$\Longrightarrow y_{p,1}(x) = 2x.$

- Look for a particular solution, $y_{p,2}(x)$, of

$$y^{(3)} + y' = -\sin x.$$

Since

$$\sin x = e^{0x}\left[P_0(x)\cos x + P_0(x)\sin x\right],$$

and $0+i$ is a root of the char. eqn. of multiplicity 1, we look for $y_{p,2}(x)$ of the form

$$y_{p,2}(x) = x^1 e^{0x}\left[Q_0(x)\cos(1\cdot x) + \tilde{Q}_0(x)\sin(1\cdot x)\right]$$

$$= x\left(B\cos x + C\sin x\right).$$

Plug $y_{p,2}(x)$ in

$$y^{(3)} + y' = -\sin x$$

to find that $B=0$, $C=\frac{1}{2}$

$\Longrightarrow y_{p,2}(x) = \frac{1}{2}x\sin x.$

Hence, the general solution of the complete nonhomogeneous equation is

$$y(x) = C_1 + C_2\cos x + C_3\sin x + 2x + \frac{1}{2}x\sin x.$$