# Mock Quiz 10

**Problem 1.** Write down the form of the partial fraction decomposition of the function

$$\frac{x+5}{(x^2-4)\,(x^2+x-6)^3\,(x^2-12x+100)}\ .$$

Do NOT determine the numerical values of the coefficients.

*Answer:* Using that $x^2-4=(x+2)(x-2)$, $x^2+x-6=(x-2)(x+3)$ (which can be found by solving the quadratic equation $x^2+x-6=0$), and $x^2-12x+100=(x-6)^2+64\geq 64>0$, we obtain that the denominator can be written as

$$(x^2-4)\,(x^2+x-6)^3\,(x^2-12x+100)=(x+2)\,(x-2)^4\,(x+3)^3\,(x^2-12x+100)\ .$$

Therefore the partial fraction decomposition has the form

$$\frac{x+5}{(x^2-4)(x^2+x-6)^3(x^2-12x+100)}=\frac{\alpha}{x+2}+\frac{\beta_1}{x-2}+\frac{\beta_2}{(x-2)^2}+\frac{\beta_3}{(x-2)^3}+\frac{\beta_4}{(x-2)^4}$$
$$+\frac{\gamma_1}{x+3}+\frac{\gamma_2}{(x+3)^2}+\frac{\gamma_3}{(x+3)^3}+\frac{Ax+B}{x^2-12x+100}\ .$$

Mathematica has a command that gives the partial fraction decomposition of a rational fraction, automatically computing the values of the unknown constants. To get the partial fraction decomposition in Mathematica, I typed

```
Apart[(x + 5)/((x^2 - 4)*(x^2 + x - 6)^3*(x^2 - 12*x + 100))]
```

then pressed `return` while holding down the `shift` key, and obtained the result

$$\frac{3}{32768}\,\frac{1}{x+2}-\frac{2663}{160000000}\,\frac{1}{x-2}+\frac{361}{8000000}\,\frac{1}{(x-2)^2}-\frac{17}{160000}\,\frac{1}{(x-2)^3}+\frac{7}{40000}\,\frac{1}{(x-2)^4}$$
$$-\frac{142719}{1905390625}\,\frac{1}{x+3}-\frac{703}{13140625}\,\frac{1}{(x+3)^2}-\frac{2}{90625}\,\frac{1}{(x+3)^3}+\frac{8287050-389171x}{62435840000000\,(x^2-12x+100)}\ .$$

Mathematica also computed the integral of this function – I typed

```
Integrate[(x + 5)/((x^2 - 4)*(x^2 + x - 6)^3*(x^2 - 12*x + 100)), x]
```

and obtained

$$\frac{1}{374615040000000}\Bigg[34297031250\log(x+2)-6234999072\log(2-x)-\frac{16904503680}{x-2}+\frac{19901424000}{(x-2)^2}$$
$$-\frac{21852544000}{(x-2)^3}-28059697152\log(x+3)+\frac{20041236480}{x+3}+\frac{4133683200}{(x+3)^2}$$
$$-1167513\log\left(x^2-12x+100\right)+4464018\tan^{-1}\left(\frac{x-6}{8}\right)\Bigg]\ .$$

Here "log" means natural logarithm, and notice that Mathematica was a bit too cavalier about the absolute values of the arguments of the logarithms.

**Problem 2.** I am using a method for computing the approximate value of definite integrals. I know that the error of this method behaves like $(\Delta x)^2$. Let $E_{10}$ be the error when I use the method by dividing the integration interval $[a, b]$ into 10 pieces (of equal length). What can you predict for the value of the error $E_{50}$ that I will have if I divide $[a, b]$ into 50 pieces?

*Answer:* When $n$ increases 5 times, $\Delta x = \frac{b-a}{n}$ will decrease 5 times, so the error, which behaves like $(\Delta x)^2$, will decrease $5^2 = 25$ times.

**Problem 3.** Will the Trapezoidal method produce a larger or a smaller value than the true value of the integral $\displaystyle\int_3^5 \sqrt{x}\, dx$?

*Answer:* Drawing a big picture of what happens in one interval $[x_{i-1}, x_i]$, and using the fact that the function $\sqrt{x}$ is increasing and concave down (because $(\sqrt{x})' > 0$ and $(\sqrt{x})'' < 0$), we can conclude that the values $I_{\text{lRs}}$, $I_{\text{rRs}}$, $I_{\text{trap}}$, $I_{\text{exact}}$ satisfy

$$I_{\text{lRs}} < I_{\text{trap}} < I_{\text{exact}} < I_{\text{rRs}} .$$

Here "lRs", "rRs", "trap" stand for the left/right Riemann sum and the trapezoidal rule; $I_{\text{lRs}}$, $I_{\text{rRs}}$, and $I_{\text{trap}}$ denote the approximate values of the integral computed by using the corresponding method, and $I_{\text{exact}}$ is the exact value of the integral.

Challenge 1: Let $E_{\text{lRs}} = |I_{\text{lRs}} - I_{\text{exact}}|$ stand for the error of the lRs method, and similarly for the other two methods. Using only geometry, can you prove that

$$E_{\text{trap}} < E_{\text{lRs}} , \quad \text{and} \quad E_{\text{rRs}} < E_{\text{lRs}} \ ?$$

How about

$$E_{\text{rRs}} + 2E_{\text{trap}} = E_{\text{lRs}} \ ?$$

(In your reasoning you have to use that the integrand is increasing and concave down.)

Challenge 2: What can you say about the value produced by the midpoint method and its error (in comparison with the errors of the other methods) applied to this integral?

Challenge 3: Can you decide if $E_{\text{trap}}$ is bigger or smaller than $E_{\text{rRs}}$ without doing any calculations?