Finding limits directly from the definition

Def: L is a limit of the sequence $\{a_n\}$ if, for every $\varepsilon > 0$, there exists a number N s.t.

$$|a_n - L| < \varepsilon \text{ for all } n > N.$$

Since it is immaterial whether the number N is an integer or not, we will <u>not</u> require N to be an integer!

Example: $\frac{\sqrt{n}}{n+1} \longrightarrow 0$. **Proof:**

$$|a_n - L| = \left| \frac{\sqrt{n}}{n+1} - 0 \right| = \frac{\sqrt{n}}{n+1} \le \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}},$$

which we require to be $< \varepsilon$:

$\frac{1}{\sqrt{n}} < \varepsilon$; this is the same as

$\frac{1}{\varepsilon^2} < n$. So we choose

$N = \frac{1}{\varepsilon^2}$. Then, for every $n > N = \frac{1}{\varepsilon^2}$,

$$|a_n - L| = \left|\frac{\sqrt{n}}{n+1} - 0\right| = \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n}$$

$$= \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\left(\frac{1}{\varepsilon}\right)} = \varepsilon. \quad \checkmark$$

$$n > N = \frac{1}{\varepsilon^2} \Rightarrow \sqrt{n} > \sqrt{N} = \frac{1}{\varepsilon^2}, \quad \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\left(\frac{1}{\varepsilon^2}\right)}$$

Example:

$$\frac{\sqrt{n} - 5}{n^3 + 7} \longrightarrow 0. \quad \text{Proof:}$$

$$|a_n - L| = \left|\frac{\sqrt{n} - 5}{n^3 + 7}\right| = \frac{|\sqrt{n} - 5|}{n^3 + 7} \underset{\text{for } \boxed{n \geq 25}}{=} \frac{\sqrt{n} - 5}{n^3 + 7}$$

$$\leq \frac{\sqrt{n}}{n^3 + 7} \leq \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}} \underset{\text{we want this!}}{<} \varepsilon$$

$\frac{1}{n^{5/2}} < \varepsilon$ is the same as

$\frac{1}{n} < \varepsilon^{2/5}$, i.e., as $n > \varepsilon^{-2/5}$.

To satisfy the last inequality, we choose

$$N = \varepsilon^{-2/5}$$

But before we also required that $n \geq 25$, so we have to impose this condition as well. Finally, we choose

$$N = \min\{25, \varepsilon^{-2/5}\}.$$

This is <u>not</u> the only possible choice! We could have, for example, continued the calculation above as follows:

$$|a_n - L| = \dots \le \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}} \le \frac{1}{n} \underset{\uparrow}{<} \varepsilon$$

we want this!

and could have set

$$N = \min\left\{25, \frac{1}{\varepsilon}\right\}$$

– this choice would work as well.

Can you think of other choices of N?

<u>Example</u>: $\frac{n^2}{n!} \longrightarrow 0$. <u>Proof</u>:

$$|a_n - 0| = \left|\frac{n^2}{n!} - 0\right| = \frac{n \cdot \cancel{n}}{1 \cdot 2 \cdot \ldots \cdot (n-2)(n-1)\cancel{n}}$$

$$\le \frac{1}{n-2} \cdot \frac{n}{n-1} \underset{\uparrow}{\le} \frac{1}{n-2} \frac{n}{0.99\, n}$$

[for those n for which $0.99n < n-1$, i.e., $1 < 0.01n$, i.e., $\boxed{n > 100}$]

$$= \frac{1}{n-2} \cdot \frac{1}{0{,}99} < \varepsilon$$

we want this

$$\Rightarrow \frac{1}{n-2} < 0{,}99\varepsilon$$

$$\Rightarrow n-2 > \frac{1}{0{,}99\varepsilon}$$

$$\Rightarrow n > 2 + \frac{1}{0{,}99\varepsilon}.$$

So we can choose

$$N = \min\left\{100,\ 2 + \frac{1}{0{,}99\varepsilon}\right\}.$$

There was no special reason in choosing $0{,}99n$ in the calculation above – instead of $0{,}99n$ we could have put $\frac{1}{2}n$ or $0{,}1n$, or any αn with $\alpha \in (0,1)$.

For example, if we had chosen

$$|a_n - 0| = \dots \leq \frac{1}{n-2} \cdot \frac{n}{n-1} \leq \frac{1}{n-2} \cdot \frac{n}{\frac{1}{2}n}$$

holds for those n for which $n-1 \geq \frac{1}{2}n$, i.e., $\frac{1}{2}n \geq 1$, i.e., $\boxed{n \geq 2}$

How should we choose N if we had made this choice?

Example: The sequence $1, 0, 1, 0, 1, 0, \ldots$ doesn't converge.

Assume that the sequence converges, i.e., that there exists a number L s.t. for every $\varepsilon > 0$ there exists N s.t.

$$|a_n - L| < \varepsilon \quad \text{for every } n > N.$$

To show that this is impossible, it is enough to find one value of $\varepsilon > 0$ for which this is impossible. We can take, for example, $\varepsilon = 0.1$. Then for every odd value of n we have

$$|1 - L| = |a_n - L| < \varepsilon = 0.1, \quad n \text{ odd},$$

while, for every even n,

$$|0 - L| = |a_n - L| < \varepsilon = 0.1, \quad n \text{ even}.$$

But the first of these inequalities means that $L \in (1 - 0.1, 1 + 0.1) = (0.9, 1.1)$,

while the second yields

$$L \in (0 - 0.1, 0 + 0.1) = (-0.1, 0.1).$$

But the intervals $(-0.1, 0.1)$ and $(0.9, 1.1)$ do not overlap, so such L doesn't exist.