Dynamical Systems and Circle Maps

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July 29, 2003

1 Introduction

We begin with an overview of dynamical systems, including the topic of circle maps. We then foray into the current research of one such type of circle map. An introduction/review of symplectic geometry is provided in preparation for the symplectic reformulation of this circle map that concludes the paper.

2 Discrete Dynamical Systems

Let

$$f^n(X) = (f \circ f \circ \ldots \circ f)(x), \tag{1}$$

i.e., f composed with itself $n-1$ times. The field of discrete dynamical systems is concerned with the question, what are the properties of the sequence $x, f(x), f^2(x), f^3(x), \ldots$? Does the limit of the sequence exist? Does the sequence converge, diverge, or become periodic?

We can use graphical analysis to visualize the answer to this question for a given value of x. We draw the graph of the function and the line $y = x$ on the same plot. Then we start at $(x, 0)$, and travel vertically to $(x, f(x))$. From there we travel horizontally to meet the line $y = x$ at the point $(f(x), f(x))$. Then we travel until we touch the graph of f at the point $(f(x), f(f(x)))$. We can continue this staircase procedure to get an idea of the limit of this sequence. In this example, $f = x^3$ and the initial x is $-.99$.

Any such sequence $f^i(x)$ with $|x| < 1$ as the starting point will converge to zero. Clearly, also, the sequences $f^i(1)$, $f^i(-1)$, and $f^i(0)$ are constant, and $\{f^i(x) | |x| > 1\}$ diverge. We can represent this in a phase portrait (Figure 2).

Phase portraits become more interesting when the function f depends on a parameter other than x. The logistic function $f(x) = rx(1-x)$ is one such well-studied function. It is often viewed as a model for population growth: beginning with a population x, the population after one time unit is a function of a parameter r that depends on the physical circumstances, multiplied by x (larger population, larger growth), multiplied by $(1-x)$, the environment being limited by too large a population. We draw the *bifurcation diagram* as r varies

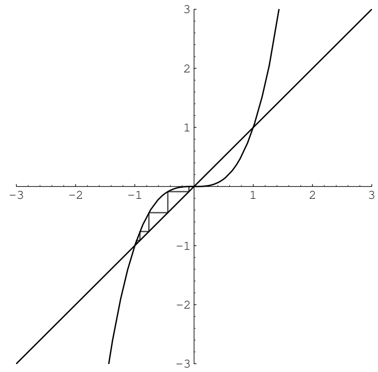


Figure 1: Graphical analysis on $y = x^3$.

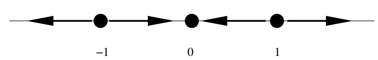


Figure 2: A phase portrait of $y = x^3$.

between 0 and 3. For each r, we pick a starting point, iterate many times, and eventually start plotting the iterates. Using this technique we can see attracting points, and attracting orbits.

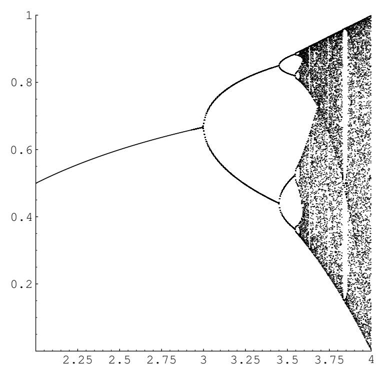


Figure 3: The bifurcation diagram for $f(x) = rx(1 - x)$.

Again there are points that attract or repel a neighborhood of points around them. Because these points are fixed under iteration, we can solve for them.

$$\begin{aligned} x &= f(x) \\ x &= rx(1-x) \\ 0 &= -rx^2 + (r-1)x \\ x &= 0, \frac{r-1}{r} \end{aligned} \tag{2}$$

Neighborhoods that are attracted to a point p are *stable sets* of p, and denoted $W^s(p)$. It appears that for $0 < r < 1$, $W^s(0) = (\frac{r-1}{r}, \infty)$. $W^s(\frac{r-1}{r}) = \frac{r-1}{r}$, and $W^s(-\infty) = (\frac{r-1}{r}, -\infty)$.

Theorem 1 (Hyperbolic Points) *Let f be a C^1 function and p a fixed point of f. Then if $|f'(p)| > 1$, there exists a neighborhood surrounding p which all points other than p must leave under iteration of f. If $|f'(p)| < 1$, then there exists a neighborhood in the stable set of p.*
p is called a hyperbolic *fixed point if $|f'(p)| \neq 1$, and a neutral fixed point if $|f'(p)| = 1$.*

The proof is an exercise in continuity and induction and is left to the reader.

From Figure 3, one can see that at $r = 3$, a period-doubling bifurcation occurs. For $x = \frac{r-1}{r}$, $f'(x) = f'(\frac{r-1}{r}) = 2 - r$, which switches from $|f'(x)| < 1$ to $|f'(x)| > 1$ at $r = 3$. Thus the fixed point becomes repelling, and the continuity of f requires that a period two attracting orbit be added. Intuitively, $f(f(x))$ has twice the degree and when $r \geq 3$, $f(f(x)) = 0$ admits twice the number of real solutions.

For larger r, we will see that there exist orbits of other periods. We introduce here Sarkovskii's Theorem, omitting the proof.

Definition 1 (Sarkovskii's Ordering) *Sarkovskii's ordering of the natural numbers:*
$3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \cdots \succ 2^2 \cdot 3 \succ 2^2 \cdot 5 \succ 2^2 \cdot 7 \succ \cdots \succ 2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \cdots \succ 2^3 \succ 2^2 \succ 2 \succ 1.$

Theorem 2 (Sarkovskii, 1964) *Suppose that f is a continuous function of the real numbers with an orbit of prime period n. Then f has orbits with prime periods of all numbers less than n in the Sarkovskii ordering.*

We now continue studying the logistic function. We make some observations about the logistic function, then delve into Cantor sets, make a formal definition of chaos, and finally conclude that we can find chaos in the logistic function. $r > 4$ will be considered, because it has simpler dynamics.

We define $\Lambda_n = \{x | h^n(x) \text{ is in } [0,1]\}$. Let $\Lambda = \bigcap_{n=1}^{\infty} \Lambda_n$.

Proposition 1 (Logistic Function) *If $f(x) = rx(1-x)$ and $r > 4$, then*

a $\{x | x \in [0,1]$ *and* $f(x) \neq [0,1]\} = (\frac{1}{2} - \frac{\sqrt{r^2-4r}}{2r}, \frac{1}{2} + \frac{\sqrt{r^2-4r}}{2r})$. *This can be seen by using the quadratic formula to solve $f(x) = 1$. Furthermore, Λ_1 is the complement of this interval in* $[0,1]$.

b The set Λ_n consists of 2^n closed intervals.

c If I is one of the closed intervals in Λ_n, then $f^n : I \to [0,1]$ is one-to-one and onto.

Definition 2 (Cantor Sets) *A Cantor set is a nonempty set $B \subset \mathbb{R}$ such that*

a B is closed and bounded, i.e., compact

b B contains no intervals

c Every $\beta \in B$ is an accumulation point of B, that is, given $\epsilon > 0$, $\forall \beta \in B, N_{\beta,\epsilon}$ contains a point in B that is not β.

The reader is probably familiar with the construction of the Cantor middle-α set. For example, to construct the middle-thirds set, one takes the unit interval, removes the middle third, then removes the middle third from the remaining two intervals, etc.

Theorem 3 (Λ and Cantor sets) *If $r > 2 + \sqrt{5}$, then Λ is a Cantor set.*

Lemma 1 *Let $r > 2 + \sqrt{5}$ and f be the logistic function. Then $\exists \lambda > 1$ such that $|f'(x)| > \lambda$ whenever $x \in \Lambda_1$. Also, the length of each interval in Λ_n is less than $(\frac{1}{\lambda})^n$*

The first part of the lemma follows from the fact that if $x \in \Lambda_1$, then $|f'(x)|$ is greater than or equal to the absolute value of the derivative of f at $(\frac{1}{2} \pm \frac{\sqrt{r^2-4r}}{2r})$. A proof of the second part can be shown using the first part together with the Mean Value Theorem.

Proof of Theorem We need to show that

a Λ is closed and bounded.
It is the intersection of closed sets, and is contained in $[0,1]$.

b Λ contains no intervals.
Suppose Λ were to contain an interval (x,y) with length $|x-y|$. Then (x,y) must be contained in an interval of all Λ_i. But we can find an n such that $|x-y| > (\frac{1}{\lambda})^n$, so that (x,y) cannot be contained in Λ_n.

c Every point in Λ is an accumulation point.
Suppose $x \in \Lambda$. Select a δ-neighborhood around x, $(x-\delta, x+\delta)$. For each n, x is contained in one of the intervals of Λ_n. We can choose n large enough that $(\frac{1}{\lambda})^n < \delta$. Then this entire interval of Λ_n is contained in the δ-neighborhood. Then both of the endpoints of this interval are in the δ-neighborhood, and at least one of them is not x. Also, all endpoints of intervals remain within Λ, since f applied to them is 0.

In a deceptively simple function, $f(x) = rx(1-x)$, we have found a Cantor set. Now we will find chaos. In mathematics, chaos is precisely defined. One such definition follows. An explanation of each clause will follow, with a proof that Λ meets the condition.

Definition 3 (Chaos) *A function $f : D \to D$ is chaotic if*

a the periodic points of f are dense in D,

b f is topologically transitive, and

c f exhibits sensitive dependence on initial conditions.

We consider $f = rx(1-x)$, and $D = \Lambda$, and $r > 2 + \sqrt{5}$.

Definition 4 (Density of Periodic Points) *If x is any point in Λ and $\epsilon > 0$, then there is a periodic point $p \in \Lambda$ such that $|x - p| < \epsilon$.*

By a previous proposition, if I is one of the closed intervals in Λ_n, then $f^n : I \to [0,1]$ is one-to-one and onto. This, together with the fixed-point theorem, implies that there are 2^n points in Λ_n with period n. These points are repelling, and, with a large enough n, are dense.

Definition 5 (Topological Transitivity) *A function $f : D \to D$ is topologically transitive if for any open sets U, V that intersect D there is a $z \in U$ and a number n such that $f^n(z) \in V \cup D$.*
Equivalently, for any $x, y \in D$, and $\epsilon > 0$, $\exists z \in D$ such that $|x - z| < \epsilon$ and $|y - f^n(z)| < \epsilon$ for some n.

Let $x, y \in \Lambda$. We will find z such that $|x - z| < \epsilon$, and $f^n(z) = y$ for some n. Choose n large enough that $(\frac{1}{\lambda})^n < \epsilon$. By previous proposition, there is an interval I_n in Λ_n such that $f^n : I_n \to [0,1]$ is 1-1 and onto. Thus some point in I_n gets taken by f^n to y, and, by our choice of n, that point is a distance less than ϵ from x.

Definition 6 (Sensitive Dependence) *A function $f : D \to D$ exhibits sensitive dependence on initial conditions if $\exists \delta > 0$ such that given any $x \in D$ and $\epsilon > 0$, there exists $y \in D$ and a number n such that $|x - y| < \epsilon$ and $|f^n(x) - f^n(y)| > \delta$.*

Let $x \in \Lambda$ and $\epsilon > 0$. Choose $\delta = \frac{1}{2}$. Again we choose n such that $(\frac{1}{\lambda})^n < \epsilon$. $x \in I_n$ for some interval in Λ_n. Recall that $f^n : I_n \to [0,1]$ is 1-1 and onto. So $\exists a, b \in I_n$ such that $f^n(a) = 0, f^n(b) = 1$. Now, $f^n(x)$ is in either $[0, \frac{1}{2})$ or $(\frac{1}{2}, 1]$, since $\frac{1}{2}$ is not in Λ. So $f^n(x)$ is a distance of at least $\frac{1}{2}$ from either $f^n(a)$ or $f^n(b)$, and we know that $|x - a| < \epsilon$ and $|x - b| < \epsilon$ by selection of n.

3 The Dynamics of Circle Maps

We provide an introduction to the dynamics, specifically, of circle maps, in preparation for our studies into the modern research of this area.

Circle maps are functions which take the circle into itself. We consider only orientation-preserving diffeomorphisms $f : S^1 \to S^1$.

It is often easier to work within $\mathbb{R}$, not S^1. We therefore define the *lift* of f.

Definition 7 (Lift) *Given a covering map* $\pi : \mathbb{R} \to S^1$,

$$\pi(x) = e^{2\pi i x} = \cos(2\pi x) + i\sin(2\pi x), \tag{3}$$

then $F : \mathbb{R} \to \mathbb{R}$ *is a lift of* f *if*

$$\pi \circ F = f \circ \pi \tag{4}$$

For example, let $f_\omega(\theta) = \theta + 2\pi\omega$, i.e., translation by an angle $2\pi\omega$. Then $F_{\omega,k} = x + \omega + k$ is a lift of f.

For any circle map, there are an infinite number of lifts; however, they always differ by an integer k. Additionally, $F(x+k) = F(x)+k$ for any integer k. This implies that $F(x+1)-(x+1) = F(x)-x$, so $F(x)-x$ is a periodic function with period 1. One can similarly show that $F^n(x) - x$ is also periodic with period 1. Using these facts, it follows that if $|x - y| < 1$, then $|F^n(x) - F^n(y)| < 1$.

Definition 8 *The* rotation number, ρ, *of a circle map measures the average rotation a point experiences under iteration of the map. It is defined as the fractional part of*

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x)}{n} \tag{5}$$

It must be shown that ρ does not depend on the choice of x. By the triangle inequality and previous remarks,

$$\begin{aligned} |F^n(x) - F^n(y)| &\le |(F^n(x) - x) - (F^n(y) - y)| + |x - y| \\ &\le 1 + |x - y| \end{aligned} \tag{6}$$

Therefore

$$\lim_{n\to\infty} \frac{|F^n(x) - F^n(y)|}{n} = 0. \tag{7}$$

So any choice of x returns the same rotation number.

Suppose we have two different lifts, F_1 and F_2. Then $\exists k \in \mathbb{N}$ such that $F_1(x) = F_2(x) + k$. Then $F_1^n(x) = F_2^n(x) + nk$, and $\rho_1(x) = \rho_2(x) + k$. Thus by taking only the fractional part of ρ_i, we have a well-defined ρ.

Theorem 4 (Periodic points and ρ) *If f has a periodic point, then $\rho(f)$ exists and is rational.*

Proof. Let $f^m(\theta) = \theta$ and $\pi(x) = \theta$. Then $F^m(x) = x + k$ for some $k \in \mathbb{N}$, and $F^{jm} = x + jk$, and we see that

$$\lim_{j\to\infty} \frac{|F^{jm}(x)|}{jm} = \lim_{j\to\infty} \frac{x+jk}{jm} = \frac{k}{m}. \tag{8}$$

Any integer n can be written as $n = jm + r, 0 \leq r < m$. Then $\exists M$ such that $|F^r(y) - y| \leq M$, $\forall y \in \mathbb{R}$, $0 \leq r < m$. From this

$$\frac{|F^n(x) - F^{jm}(x)|}{n} = \frac{|F^r(F^{jm}(x)) - F^{jm}(x)|}{n} \leq \frac{M}{n}, \tag{9}$$

and when we take n to infinity, the limits are equal. The proof for f without periodic points is more complicated; but again, ρ is well defined. We shall, however, show the following.

Theorem 5 (No periodic points and ρ) *$\rho(f) \notin \mathbb{Q}$ if and only if f has no periodic points*

Proof. Given the previous theorem, it suffices to show that if f has no periodic points, then ρ is irrational. We assume ρ is rational and derive a contradiction. Let $\rho_0(F) = \lim_{n\to\infty} \frac{F^n(x)}{n}$, i.e., ρ without the integer part omitted. Then for any lift F, $\rho_0(F^m) = m\rho_0(F)$. Thus we may assume that $\rho(F) = 0$, but f has no fixed points. Since F also has no fixed points, we may assume $F(x) > x\,\forall x$. We must consider the two cases of $F^n(0) < 1\,\forall n$, or $\exists k > 0 | F^k(0) > 1$. In this latter case $F^{mk}(0) > m \Rightarrow \rho_0(F) > \frac{1}{k}$, which is a contradiction. In the first case $F^n(0)$ is monotonically increasing in $[0,1]$ and therefore must converge. This limit point is a fixed point of F.

As we did in the previous section, we turn now to an important specific map, and find a Cantor function.

We consider the map

$$f_{\omega,\epsilon}(\theta) = \theta + 2\pi\omega + \epsilon\sin(\theta) \tag{10}$$

with the lift

$$F_{\omega,\epsilon}(x) = x + \omega + \frac{\epsilon}{2\pi}\sin(2\pi x). \tag{11}$$

For $\epsilon = 0$, this is the rotation map.
For $0 \leq \epsilon < 1$, $f_{\omega,\epsilon}$ is a diffeormorphism of S^1.
For $\epsilon = 1$, the map is a homeomorphism, and for $\epsilon > 1$, it is no longer injective.

If $\omega_1 > \omega_2$, then

$$F_{\omega_1,\epsilon}(x) > F_{\omega_2,\epsilon}(x) \tag{12}$$

for all $x \in \mathbb{R}$. Also,

$$F^n_{\omega_1,\epsilon}(x) > F^n_{\omega_2,\epsilon}(x) \tag{13}$$

for all n, and so $\rho_0(F_{\omega_1,\epsilon}) \geq \rho_0(F_{\omega_2,\epsilon})$. Thus ρ_0 is a nondecreasing function of ω; it can also be shown that it is continuous. Let us fix $\epsilon \neq 0$ and denote $f_{\omega,\epsilon}$ as f_ω. The graph of ρ versus ω looks as follows:

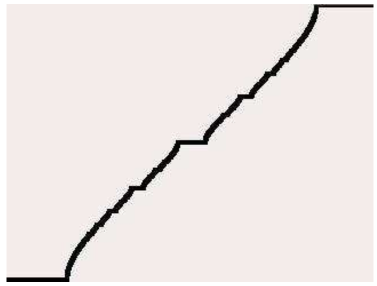


Figure 4: The graph of $\rho(f_{\omega_0})$ is also known as the devil's staircase.

Theorem 6 *For any rational rotation number $\frac{p}{q}$, there is an interval of ω-values for which the rotation number remains $\frac{p}{q}$.*

Proof. Suppose that $\rho(f_{\omega_0})$ is rational, of the form $\frac{p}{q}$. Then f_{ω_0} has a period q point. There exists $x_0 \in \mathbb{R}$ such that

$$F^q_{\omega_0}(x_0) = x_0 + k, \text{ with } k = p \tag{14}$$

Consider the graph of $y = F^q_{\omega_0}(x)$. This graph intersects the straight line $y = x + k$ at the point $(x_0, x_0 + k)$. If $F^q_{\omega_0}{}' \neq 1$, then the Implicit Function Therem tells us that there is an open interval about ω_0 for which the graphs of each F^q_ω also intersect the line $y = x + k$. If the derivative is 1, then we can use the fact that F^q is analytic to find an integer j for which $(F^q_{\omega_0})^{(j)}(x_0) \neq 0$. If j is odd, then the graphs of nearby F^q_ω must pierce the line $y = x + k$. If j is even, then either $F^q_{\omega_0}$ is either concave up or concave down, and again nearby F^q_ω must intersect the line $y = x + k$.

There is a unique ω for which $\rho(f_\omega)$ is a given irrational number.

The graph of $\rho(f_\omega)$ is a Cantor function. It is constant everywhere but on a Cantor set.

4 Research into Circle Maps

The following comes from the article *Off-Center Reflections: Caustics and Chaos*, by Thomas Kwok-keung Au and Xiao-Song Lin.

We consider off-center reflection maps. To construct such a map, $R_r : S^1 \to S^1$, we pick a point $(r, 0)$ in the interior of the unit disk D^2. Then for every $\phi \in s^1$, we emit a ray from $(r, 0)$ to ϕ, and reflect ϕ by the angle made with the origin to $R_r(\phi)$.

We have

$$R_r(\phi) = \phi + \pi - 2\alpha \mod 2\pi \qquad \text{where } \alpha = \alpha(\phi) := \operatorname{Arg}(\cos\phi - r + i\sin\phi) - \phi, \tag{15}$$

recalling that $\operatorname{Arg}(e^{i\theta}) = \theta$.

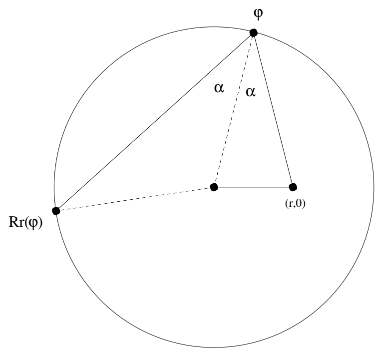


Figure 5: The off-center reflection map.

We can get another equation for the map R_r by using complex numbers with modulus 1:

$$R_r(z) = -z^2 \frac{1 - rz}{z - r} \tag{16}$$

For future reference, we put here the first two derivatives of R_r.

$$R_r'(\phi) = \frac{1 - 4r\cos\phi + 3r^2}{1 - 2r\cos\phi + r^2} \tag{17}$$

$$R_r''(\phi) = \frac{2r(1 - r^2)\sin\phi}{(1 - 2r\cos\phi + r^2)^2} \tag{18}$$

These can be calculated by expressing α in terms of the arctangent.

For any circle map f, including R_r, we get a family of lines from ϕ to $f(\phi)$. This family is

$$\begin{aligned} F(\phi, x, y) &= (\sin f(\phi) - \sin\phi)(x - \cos\phi) - (\cos f(\phi) - \cos\phi)(y - \sin\phi) \\ &= (\sin f(\phi) - \sin\phi)x - (\cos f(\phi) - \cos\phi)y - \sin(f(\phi) - \phi). \end{aligned} \tag{19}$$

Definition 9 (Envelope and Caustic) *The envelope of a family of lines is the curve which touches every one of those lines. The caustic of the map f is the envelope of these lines. The caustic can be found by solving the equations*

$$\frac{\partial F}{\partial \phi}(\phi, x, y) = 0 = F(\phi, x, y) \tag{20}$$

For example, consider the family of lines shown in the following diagram.

This family is described by

$$F(\alpha, x, y) = x + y\tan(\alpha) - \frac{1}{\cos(\alpha)} \tag{21}$$

with derivative

$$\frac{\partial}{\partial \alpha} F(\alpha, x, y) = \frac{y}{\cos^2(\alpha)} - \frac{\sin(\alpha)}{\cos^2(\alpha)} \tag{22}$$

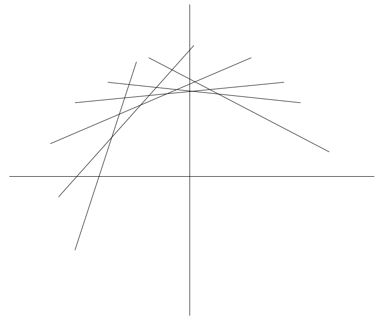


Figure 6: A family of lines.

We set $F(\alpha, x, y) = 0 = \frac{\partial}{\partial\alpha}F(\alpha, x, y)$ Multiplying (22) through by $\cos^2(\alpha)$, we get $y = \sin(\alpha)$. Thus $\cos(\alpha) = \sqrt{1-\sin^2(\alpha)} = \sqrt{1-y^2}$. We substitute these terms into (21), to get $x + y\frac{y}{\sqrt{1-y^2}} - \frac{1}{\sqrt{1-y^2}} = 0$, so $x = \sqrt{1-y^2}$, and, as you may have guessed, we have the unit circle.

Definition 10 (Cusp) *A cusp is a point on a continuous curve where the tangent vector changes sign. For example:*

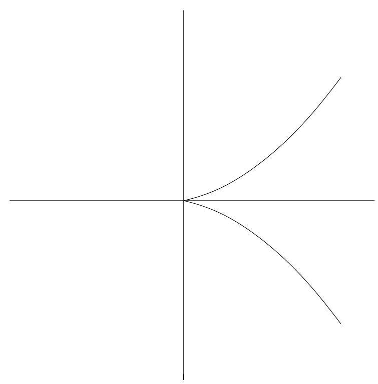


Figure 7: A cusp.

To solve for the caustic of a family of lines created by a map, (19) is used. F is the first row of the following matrix; $\frac{\partial F}{\partial\phi}$ is the second.

$$\begin{pmatrix} \sin f(\phi) - \sin\phi & -\cos f(\phi) + \cos\phi \\ f'(\phi)\cos f(\phi) - \cos\phi & f'(\phi)\sin f(\phi) - \sin\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin(f(\phi) - \phi) \\ (f'(\phi) - 1)\cos(f(\phi) - \phi) \end{pmatrix}$$

By solving for x and y, we obtain a parameterization of the caustic:

$$x(\phi) = \frac{f'(\phi)\cos\phi + \cos f(\phi)}{1 + f'(\phi)} \tag{23}$$

$$y(\phi) = \frac{f'(\phi)\sin\phi + \sin f(\phi)}{1 + f'(\phi)} \tag{24}$$

By simple calculation, the derivatives of x and y are

$$x'(\phi) = \frac{f''(\phi)(\cos\phi - \cos f(\phi)) - f'(\phi)(1 + f'(\phi))(\sin\phi + \sin f(\phi))}{(1 + f'(\phi))^2} \tag{25}$$

$$y'(\phi) = \frac{f''(\phi)(\sin\phi - \sin f(\phi)) - f'(\phi)(1 + f'(\phi))(\cos\phi + \cos f(\phi))}{(1 + f'(\phi))^2} \tag{26}$$

Theorem 7 (Cusp points) *For all $0 < r < 1$, there are exactly four cusp points on the caustic of R_r.*

Proof. Express the derivatives of x and y in terms of r and ϕ, using (17) and (25).

$$x'(\phi) = \frac{6r^2(-\cos\phi + r\cos(2\phi))(r - \cos\phi)\sin\phi}{(-1 - 2r^2 + 3r\cos\phi)^2} \tag{27}$$

$$y'(\phi) = \frac{6r^2(-1 + 2r\cos\phi)(r - \cos\phi)\sin^2\phi}{(-1 - 2r^2 + 3r\cos\phi)^2} \tag{28}$$

The solutions for $x'(\phi) = 0 = y'(\phi)$ are $\phi = 0, \pi$ and two values of ϕ with $\cos\phi = r$.

A few diagrams are in order to show what the caustic looks like.

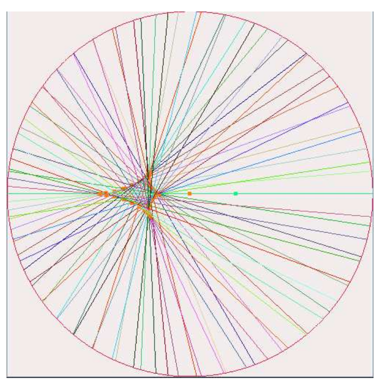


Figure 8: For $r \sim .25$, the lines between ϕ and $R_r(\phi)$ are graphed for fifty values of ϕ. Points on the caustic are boxed.

The caustic runs to infinity iff $r \geq \frac{1}{2}$.

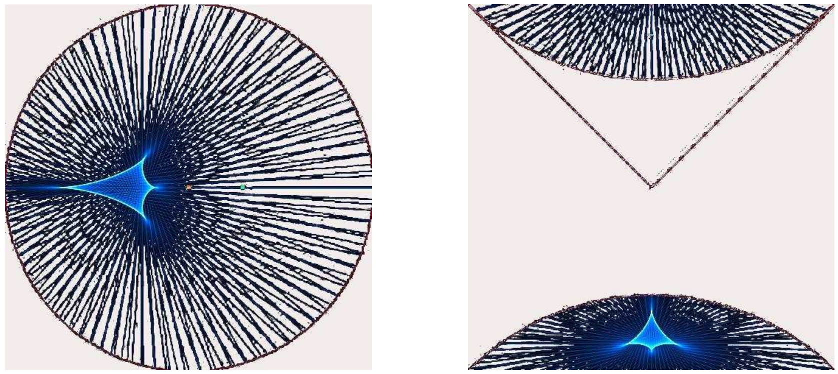


Figure 9: The caustic is shown here with $r \sim .25$.

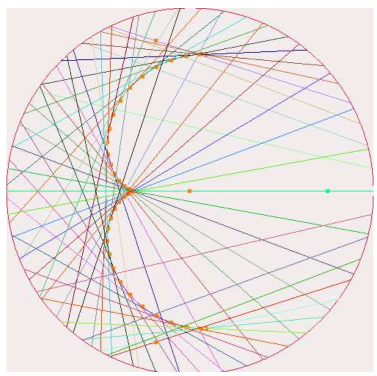


Figure 10: For $r \sim .75$, the lines between ϕ and $R_r(\phi)$ are graphed for fifty values of ϕ. Points on the caustic are boxed.

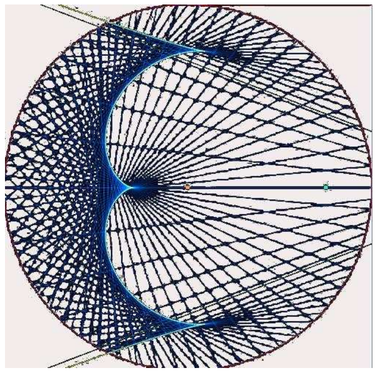
As one can see, further investigations in this manner would become very computation-intensive. A symplectic approach, surprisingly, overcomes this difficulty. We will now give a quick review of manifolds and forms, and a short review/introduction to symplectic geometry.

5 Symplectic Geometry

Definition 11 (Basic terminology) *A manifold M of dimension n can be locally transformed, through coordinate charts, to open balls in $\mathbb{R}^n$.*

The tangent space at a point of the manifold, T_xM, consists of all vectors from the point x into the copy of $\mathbb{R}^n$ tangent to M at that point.

The cotangent space at a point, T^*_xM, is the set of all dual vectors to the tangent space at x - i.e., it would have basis $\{dx, dy, dz\}$ if M had dimension 3.



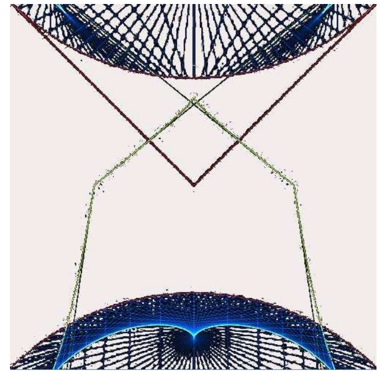


Figure 11: The caustic is shown here with $r \sim .25$. In the figure on the left, we cannot see the part of the caustic that runs to positive infinity; it is already off the diagram. Thus we have constructed the strange perspective of the figure on the right. Here 0 is at the bottom, we head upwards to where $-\infty$ meets $+\infty$ in the middle, then decrease to 0 again at the top. Two parallel lines are drawn to help with perspective - they meet at infinity! We can see that the parts of the caustic that run to $\pm\infty$ meet nicely.

The cotangent bundle, denoted T^*M, consists of the set $\{(q,p) : q \in M, p \in \mathbb{R}^n\}$. It has dimension $2n$. The one-forms in the cotangent space, in their most general form, look like $\alpha_i dq^i + \beta_i dp^i$.

A fiber is the preimage, under a map, of a point in the manifold.

A section of a bundle is a preimage of the entire manifold, under some map, into the bundle.

Nondegerate means that if $\omega(x, \cdot) = 0$ always, then x must be zero.

Definition 12 (Symplectic manifold) *A symplectic manifold is a smooth manifold with a nondegenerate two-form ω, called its symplectic structure. Let (M, ω) denote a symplectic manifold.*

A symplectic manifold always has even dimension.

For example, the cotangent bundle has a natural symplectic structure, with $\omega = dq^i \wedge dp^i$.

Definition 13 (Lagrangian manifold) *An n-dimensional submanifold L of a symplectic manifold is Lagrangian if the restriction of ω to L vanishes (is zero everywhere).*

Theorem 8 (Darboux Coordinates) *There exist coordinates, called Darboux coordinates, $(q^1, p^1, q^2, p^2, \ldots, q^n, p^n)$ that make the non-degenerate two-form associated with a symplectic manifold look like $\omega = \sum_i dq^i \wedge dp^i$.*

(The following proofs will lapse into easier vector space notation; manifolds are locally isomorphic to $\mathbb{R}^n$, a vector space, so this is admissible.)

Definition 14 (ω-orthogonal complement) *Consider a symplectic vector space (V, ω) and a subspace W of V. We define the* ω-orthogonal complement *of W in V as follows:*

$$W^{\omega} = \{v \in V | \forall w \in W : \omega(v, w) = 0\}. \tag{29}$$

If the intersection of W and W^{ω} is trivial, then quite clearly W is a symplectic subspace. dim $W^{\omega}+$ dim $W =$ dim V. Additionally, W^{ω} is also symplectic. By our previous description of Lagrangian, we see that W is Lagrangian if $W = W^{\omega}$, and the dimension formula confirms that dim W must be dim $V/2$.

Proof of theorem. We construct a basis for (V, ω), $\{q_1, p_1, q_2, p_2, \ldots, q_n, p_n\}$ such that $\omega(q_j, p_k) = \delta_{jk}$, $\omega(q_j, q_k) = \omega(p_j, p_k) = 0$.
Pick a nonzero vector q_1. Then, since ω is nondegenerate, we can find a vector p_1 such that $\omega(q_1, p_1) = 1$. Call the subspace spanned by $\{q_1, p_1\}$, V_1. Clearly V_1 is symplectic, and by the previous paragraph we have $V = V_1 \oplus V_1^{\omega}$, where V is the direct sum of 2 symplectic manifolds. We can continue to operate inductively on V_1^{ω}, cutting off two vectors at a time, until we have constructed the desired basis.

Definition 15 (Standard symplectic form) *We can now define the standard symplectic form. If we let $V = \mathbb{R}^{2n}$ and $\{q_1, p_1, q_2, p_2, \ldots, q_n, p_q\}$ be a basis of V constructed as above, then, with*

$$v = \sum_{j=1}^{n}(a_j q_j + b_j p_j) \text{ and } v' = \sum_{j=1}^{n} a'_j q_j + b'_j p_j \tag{30}$$

it follows naturally that

$$\omega(v, v') = \sum_{j=1}^{n}(a_j b'_j - a'_j b_j) \tag{31}$$

For example, fibers of a cotangent bundle are Lagrangian manifolds. $T_x^*M = \{X^i \frac{\partial}{\partial q^i} + Y^j \frac{\partial}{\partial p^j}\}$. The q^i are coordinates in the base manifold; the p^j are coordinates in the fiber. $\omega = dq^i \wedge dp^j$. Since we are selecting fibers, we use only the p^j, so a member of the fiber looks like $Z \frac{\partial}{\partial p^j}$. $\omega(Z_1 \frac{\partial}{\partial p^j}, Z_2 \frac{\partial}{\partial p^j}) = 0$.

Another example: Given a smooth function on a manifold B with dimension n, the graph of its differential, considered as a section of the cotangent bundle, is a Lagrangian manifold in T^*B. Let $f : B \to \mathbb{R}$. $df = \frac{\partial f}{\partial q^i} dq^i$, a one-form. $\omega = dq^i \wedge dp^i$. $\omega(df, dg) = 0$.

Consider a function on a symplectic manifold M, $H : M \to \mathbb{R}$, the Hamiltonian function. It naturally defines a vector field, called the Hamiltonian vector field, X_H, where $X_H(q^1, p^1, \ldots, q^n, p^n) \in T_{(q^1, p^1, \ldots, q^n, p^n)}M$. We will write a vector in X_H as $A^i \frac{\partial}{\partial q^i} + B^j \frac{\partial}{\partial p^j}$. We can use ω and X_H to define an isomorphism between the tangent and cotangent bundles. We want $\omega(X_H, \cdot) = dH$. We are using ω to change a vector in the tangent space into a 1-form. Recall $dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p^j} dp^j$.

$$\omega(X_H, \cdot) = \sum_i (dq^i \otimes dp^i - dp^i \otimes dq^i)(A^i \frac{\partial}{\partial q^i} + B^j \frac{\partial}{\partial p^j}, \cdot) \qquad (32)$$
$$= A^i dp^i - B^i dq^i$$

So, to make the right-hand side equal dH, we see that $A^i = \frac{\partial H}{\partial p^j}$ and $B^i = -\frac{\partial H}{\partial q^i}$. Thus $X_H = \frac{\partial H}{\partial p^i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p^i}$.

6 The Dynamics of Circle Maps - Symplectic Reformulation

We denote the coordinates of the unit cotangent bundle $ST^*(\mathbb{R}^2)$ by (p_x, p_y, x, y), where $x, y \in \mathbb{R}^2$ and $p_x^2 + p_y^2 = 1$. This bundle is a 3-dimensional manifold with the 1-form $p_x dx + p_y dy$. The cotangent manifold $T^*(\mathbb{R}^2)$ is symplectic with the symplectic 2-form $d(p_x dx + p_y dy)$.

The unit vector (p_x, p_y) in the direction from ϕ to $R_r(\phi)$ is as follows, and can be visualized with Figure 12.

$$p_x = \cos(\phi + \pi - \alpha), \qquad p_y = \sin(\phi + \pi - \alpha). \qquad (33)$$

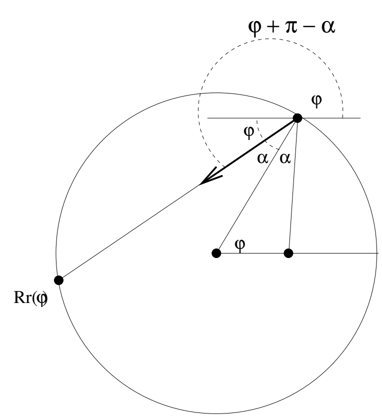


Figure 12: The unit vector located at the point ϕ.

Consider the map L:

$$\begin{pmatrix} \phi \\ S \end{pmatrix} \mapsto \begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\phi + \pi - \alpha) \\ \sin(\phi + \pi - \alpha) \\ \cos(\phi) + S\cos(\phi + \pi - \alpha) \\ \sin(\phi) + S\sin(\phi + \pi - \alpha) \end{pmatrix}$$

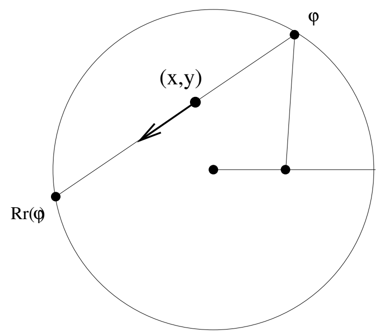


Figure 13: The point (x, y) with the tangent vector.

Graphically, depending on S and ϕ, we map (x, y) to a point somewhere along the line between ϕ and $R_r(\phi)$ (Figure 13).

The map $L : S^1 \times \mathbb{R}^1 \to T^*(\mathbb{R}^2)$ can be thought of as a flow in the parameter S of unit speed in the direction of the reflection lines.

Let $p : T^*(\mathbb{R}^2) \to \mathbb{R}^2$ be the canonical projection, i.e.,

$$p : \begin{pmatrix} p_x \\ p_y \\ x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix}$$

The Jacobian $J(p \circ L)$ is (taking $\frac{\partial}{\partial S}, \frac{\partial}{\partial \phi}$):

$$\det \begin{pmatrix} \cos(\phi + \pi - \alpha) & -\sin(\phi) - S(1-\alpha')\sin(\phi + \pi - \alpha) \\ \sin(\phi + \pi - \alpha) & \cos(\phi) + S(1-\alpha')\cos(\phi + \pi - \alpha) \end{pmatrix} = -\cos(\alpha) + S(1-\alpha')$$

Setting the Jacobian to zero, we get the equation for the *critical curve.*

$$S = \frac{\cos(\alpha)}{1 - \alpha'} \tag{34}$$

Proposition 2 *The critical curve, when mapped to the (x, y) plane, is the caustic of R_r.*

Intuitively, why is this so? First let us get a deeper understanding of the caustic. ϕ, x, and y determine a two dimensional surface embedded in $\mathbb{R}^3$. The caustic is the projection onto $\mathbb{R}^2$ of the points at which $\frac{\partial}{\partial \phi} = 0$, i.e., where one can travel in the ϕ direction and x and y will not change. A few perspectives of a 3-D diagram will illustrate this (Figure 14).

When the Jacobian of $p \circ L$ is zero, we are mapping very small $\phi \times S$ rectangle-neighborhoods, not into warped rectangles, but into something degenerate - a line, or a point. Along the caustic, by our previous definition, $\frac{\partial}{\partial \phi} = 0$ on these neighborhoods, so we have shrunk our $\phi \times S$ rectangles to lines with no ϕ-width.

We can further use the symplectic structure we have developed.

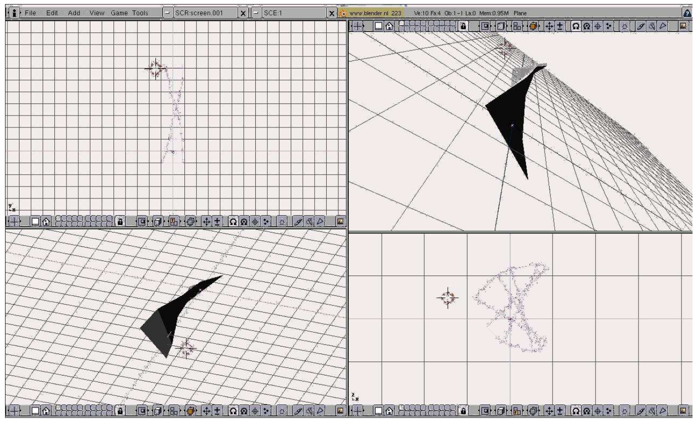


Figure 14: Four views of a twisted plane, drawn in Blender. The top left panel gives a clear view of the figure. When the horizontal lines in the surface are viewed from above, projected downward, we get the image in the bottom right panel - one can clearly see the caustic.

Theorem 9 *We have the following equality:*

$$p_x dx + p_y dy = \sin \alpha d\phi + dS. \tag{35}$$

Proof.

$$\begin{aligned}
d_x &= d(\cos\phi + S\cos(\phi + \pi - \alpha)) \\
&= \cos(\phi + \pi - \alpha)dS + (-\sin\phi - S(1-\alpha')\sin(\phi + \pi - \alpha))d\phi \\
d_y &= d(\sin\phi + S\sin(\phi + \pi - \alpha)) \\
&= \sin(\phi + \pi - \alpha)dS + (\cos\phi - S(1-\alpha')\cos(\phi + \pi - \alpha))d\phi \\
p_x dx + p_y dy &= (\cos^2(\phi + \pi - \alpha) + \sin^2(\phi + \pi - \alpha))dS \\
&\quad + [(\sin(\phi + \pi - \alpha)\cos\phi - cos(\phi + \pi - \alpha)\sin\phi) \\
&\quad + S(1-\alpha')(\cos(\phi + \pi - \alpha)\sin(\phi + \pi - \alpha) \\
&\qquad - \sin(\phi + \pi - \alpha)\cos(\phi + \pi - \alpha))]d\phi \\
p_x dx + p_y dy &= dS + \sin(\phi + \pi - \alpha - \phi)d\phi \\
p_x dx + p_y dy &= dS + \sin\alpha d\phi
\end{aligned} \tag{36}$$

Taking the exterior differential of both sides, $d(\sin\alpha d\phi + dS) = 0$, so the image of L is a Lagrangian cylinder in $T^*(\mathbb{R}^2)$. Since α is an odd function of ϕ, we know that

$$\int_{S^1} \sin\alpha d\phi = 0 \tag{37}$$

so we have an *exact* Lagrangian cylinder. To get a section of this cylinder, we can use the two-form to define a function $S(\phi)$ on the circle by

$$S(\phi) = -\int_0^{\phi} \sin \alpha d\phi. \tag{38}$$

The properties of S can be used to study the shape of the caustic, including the shape of the caustic formed by iterations of $R_r(\phi)$.

References

[1] T. Au, X. Lin, "Off Center Reflections: Caustics and Chaos", *Experimental Mathematics* **10**:2 (2001), 287-301.

[2] R. Devaney, "An Introduction to Chaotic Dynamical Systems", *Addison-Wesley*, 1989.

[3] R. Holmgren, "A First Course in Discrete Dynamical Systems", *Springer*, New York, 1996.

[4] E. Weisstein, http://www.mathworld.com, *CRC Press LLC*, 1999, *Wolfram Research, Inc.*, 1999-2003.

Thank you to Professor Nikola Petrov for guiding my research and to Jim McCann for C++ help.

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