Volume of a ball of radius R in $\mathbb{R}^n$

The ball of radius R in $\mathbb{R}^n$ is defined as

$$B_n(R) := \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq R^2\} \ ,$$

and the sphere of radius B in $\mathbb{R}^n$ is its boundary:

$$S_n(R) := \partial B_n(R) = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 = R^2\} \ .$$

Let $V_n(R)$ and $A_n(R)$ stand for the volume of $B_n(R)$ and the area of $S_n(R)$, respectively:

$$V_n(R) := \text{Volume of } B_n(R) \ , \qquad A_n(R) := \text{Area of } S_n(R) \ .$$

Observation: Just by looking at the units of volume and area in $\mathbb{R}^n$ – which are $(\text{length})^n$ and $(\text{length})^{n-1}$, respectively – we see that

$$V_n(R) = V_n(1)\, R^n \ , \qquad A_n(R) = A_n(1)\, R^{n-1} \ . \tag{1}$$

Observation: If the radius R of a ball in $\mathbb{R}^n$ increases by a small amount h, the volume of the ball increases approximately by the area of the surface of the ball multiplied by h, which implies that

$$V_n'(R) = \lim_{h\to 0} \frac{V_n(R+h) - V_n(R)}{h} \approx \lim_{h\to 0} \frac{A_n(R)\, h}{h} = A_n(R) \ ,$$

hence

$$V_n'(R) = A_n(R) \tag{2}$$

Since $V_n(0) = 0$, this implies that

$$V_n(R) = \int_0^R A_n(\rho)\, \mathrm{d}\rho \ . \tag{3}$$

More generally, if $f : \mathbb{R}^n \to \mathbb{R}$ is a function that depends only on the distance $|\mathbf{x}|$ from $\mathbf{x}$ to the origin $\mathbf{0} \in \mathbb{R}^n$, i.e., $f(\mathbf{x}) = \tilde{f}(|\mathbf{x}|)$, the same reasoning yields

$$\int \cdots \int_{\mathbb{R}^n} f(\mathbf{x})\, \mathrm{d}V = \int_0^R \tilde{f}(\rho)\, A_n(\rho)\, \mathrm{d}\rho \ , \qquad \rho := |\mathbf{x}| \ . \tag{4}$$

Computation of $V_n(1)$: Because of (1), it is clear that to find $V_n(R)$, it is enough to compute the volume $V_n(1)$ of the unit ball in $\mathbb{R}^n$. One can obtain this by computing the value of the integral

$$J_a := \int \cdots \int_{\mathbb{R}^n} \mathrm{e}^{-a|\mathbf{x}^2|}\, \mathrm{d}V \tag{5}$$

in two different ways. Here a is a positive constant, $\mathrm{d}V$ is the volume element in $\mathbb{R}^n$, and the integration is performed over the whole space $\mathbb{R}^n$.

First way of computing J_a: The integral in the definition (5) of J_a is a product of n one-dimensional integrals:

$$
\begin{aligned}
J_a &= \int \cdots \int_{\mathbb{R}^n} \mathrm{e}^{-a|\mathbf{x}^2|}\,\mathrm{d}V = \int_{\mathbb{R}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathrm{e}^{-a(x_1^2+x_2^2+\cdots+x_n^2)}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\mathrm{d}x_n \\
&= \int_{\mathbb{R}} \mathrm{e}^{-ax_1^2}\,\mathrm{d}x_1 \int_{\mathbb{R}} \mathrm{e}^{-ax_2^2}\,\mathrm{d}x_2 \cdots \int_{\mathbb{R}} \mathrm{e}^{-ax_n^2}\,\mathrm{d}x_n \\
&= \left(\int_{\mathbb{R}} \mathrm{e}^{-ax^2}\,\mathrm{d}x\right)^n =: (I_a)^n \ .
\end{aligned}
$$

To compute I_a, we use polar coordinates in $\mathbb{R}^2$ in the expression for $(I_a)^2$:

$$
\begin{aligned}
(I_a)^2 &= \left(\int_{\mathbb{R}} \mathrm{e}^{-ax^2}\,\mathrm{d}x\right)\left(\int_{\mathbb{R}} \mathrm{e}^{-ay^2}\,\mathrm{d}y\right) = \iint_{\mathbb{R}^2} \mathrm{e}^{-a(x^2+y^2)}\,\mathrm{d}A \\
&= \int_0^{2\pi}\int_0^{\infty} \mathrm{e}^{-ar^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta = 2\pi \int_0^{\infty} \mathrm{e}^{-ar^2}\,r\,\mathrm{d}r = \frac{\pi}{a}\int_0^{\infty} \mathrm{e}^{-\xi}\,\mathrm{d}\xi = \frac{\pi}{a}
\end{aligned}
\tag{6}
$$

(with the substitution $\xi := ar^2$). This implies that $I_a = \sqrt{\frac{\pi}{a}}$, and

$$
J_a = \left(\frac{\pi}{a}\right)^{n/2} \ . \tag{7}
$$

Second way of computing J_a: From (2) and (1), we see that

$$
A_n(R) = V_n'(R) = \frac{\mathrm{d}}{\mathrm{d}R}\left[V_n(1)\,R^n\right] = nV_n(1)\,R^{n-1} \ .
$$

This observation and (4) allow us to write

$$
\begin{aligned}
J_a &= \int \cdots \int_{\mathbb{R}^n} \mathrm{e}^{-a|\mathbf{x}^2|}\,\mathrm{d}V = \int_0^{\infty} \mathrm{e}^{-a\rho^2}\,A_n(\rho)\,\mathrm{d}\rho \\
&= nV_n(1)\int_0^{\infty} \mathrm{e}^{-a\rho^2}\,\rho^{n-1}\,\mathrm{d}\rho = nV_n(1)\,\tfrac{1}{2}\,a^{-\frac{n}{2}}\int_0^{\infty} \mathrm{e}^{-t}\,t^{\frac{n}{2}-1}\,\mathrm{d}t \\
&= V_n(1)\,\frac{n}{2}\,\Gamma\left(\frac{n}{2}\right)a^{-\frac{n}{2}} = V_n(1)\,\Gamma\left(\frac{n}{2}+1\right)a^{-\frac{n}{2}}
\end{aligned}
\tag{8}
$$

(in computing the value of the integral, we used the substitution $t := a\rho^2$). Here we used the definition (9) and the property (10) of the Gamma function.

Putting everything together: Comparing the right-hand sides of (7) and (8), we see that

$$V_n(1) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\ ,$$

therefore the volume of the n-dimensional ball of radius R is

$$V_n(R) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\, R^n\ .$$

Check that this works in the well-known cases: $V_1(R) = 2R$, $V_2(R) = \pi R^2$, $V_3(R) = \frac{4}{3}\pi R^3$.

Gamma function

Definition:

$$\Gamma(x) := \int_0^\infty \mathrm{e}^{-t}\, t^{x-1}\, \mathrm{d}t \qquad \text{for } x > 0\ . \tag{9}$$

Basic property:

$$\Gamma(x+1) = x\,\Gamma(x) \qquad \text{for } x > 0\ , \tag{10}$$

which can be obtained directly by setting $u := t^x$ and $v := \mathrm{e}^{-t}$ and integrating by parts:

$$\Gamma(x+1) = \int_0^\infty \mathrm{e}^{-t}\, t^x\, \mathrm{d}t = -\int_{t=0}^\infty u\, \mathrm{d}v = -uv\big|_{t=0}^\infty + \int_{t=0}^\infty v\, \mathrm{d}u$$

$$= -\left(t^x \mathrm{e}^{-t}\right)\big|_{t=0}^\infty + \int_{t=0}^\infty \mathrm{e}^{-t}\, x t^{x-1}\, \mathrm{d}t = x \int_{t=0}^\infty \mathrm{e}^{-t}\, t^{x-1}\, \mathrm{d}t = x\,\Gamma(x)\ .$$

Particular values

(a) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \mathrm{e}^{-t}\, t^{-1/2}\, \mathrm{d}t = 2\int_0^\infty \mathrm{e}^{-s^2}\, \mathrm{d}s = \int_{-\infty}^\infty \mathrm{e}^{-s^2}\, \mathrm{d}s = I_1 = \sqrt{\pi}$ (recall (6)).

(b) $\Gamma(1) = \int_0^\infty \mathrm{e}^{-t}\, \mathrm{d}t = 1$.

(c) For any integer $m \geq 1$, using (10) and the value of $\Gamma(1)$, we obtain

$$\Gamma(m) = (m-1)\Gamma(m-1) = (m-1)(m-2)\Gamma(m-2)$$

$$= \cdots = (m-1)(m-2)\cdots(2)(1)\,\Gamma(1) = (m-1)!\ .$$

(d) For any integer $m \geq 0$, using (10) and the value of $\Gamma(\frac{1}{2})$, we obtain

$$\Gamma(m+\tfrac{1}{2}) = (m-\tfrac{1}{2})\,\Gamma(m-\tfrac{1}{2}) = (m-\tfrac{1}{2})(m-\tfrac{3}{2})\,\Gamma(m-\tfrac{3}{2}) = \cdots$$

$$= (m-\tfrac{1}{2})(m-\tfrac{3}{2})\cdots(\tfrac{3}{2})(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}) = \tfrac{(2m-1)(2m-3)\cdots(3)(1)}{2^m}\sqrt{\pi}$$

$$= \tfrac{(2m)(2m-1)(2m-2)(2m-3)(2m-4)\cdots(4)(3)(2)(1)}{2^m\,[2m][2(m-1)]\cdots[2\cdot 2][2\cdot 1]}\sqrt{\pi} = \frac{(2m)!}{4^m\, m!}\sqrt{\pi}\ .$$