CALCULUS III FALL 1999
HOMEWORK 3 – ANSWERS

§9.5 Questions 2,8,12,32,38; §9.7 Questions 4,6,16,24

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2. The required area is

$$A = \frac{1}{2}\int_{-\pi/2}^{\pi/2} e^{2\theta}\,d\theta = \frac{1}{4}\left[e^{2\theta}\right]_{-\pi/2}^{\pi/2} = \frac{1}{4}\left(e^{\pi} - e^{-\pi}\right).$$

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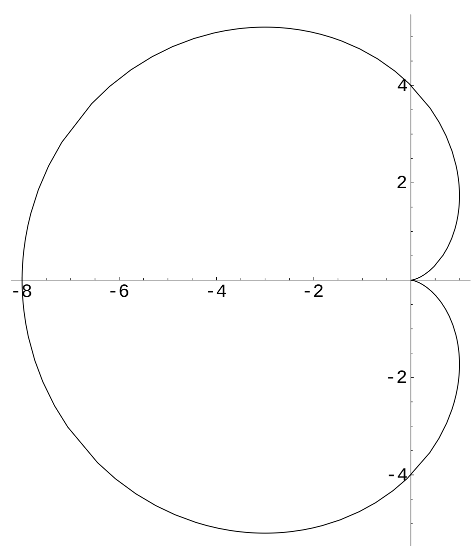
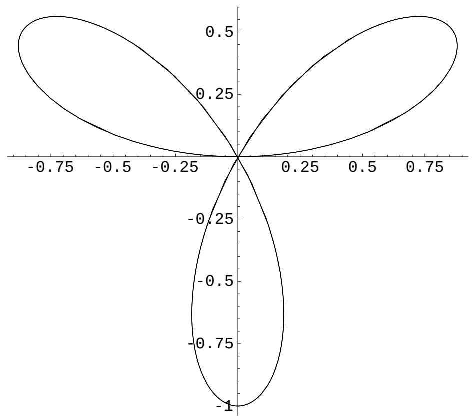
8. The curve is the limaçon shown below. It is described once as θ varies from 0 to 2π.



The area it encloses is

$$A = \frac{1}{2}\int_0^{2\pi} 16(1-\cos\theta)^2\,d\theta = 8\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta = 8\int_0^{2\pi}\frac{3}{2}\,d\theta = 24\pi.$$

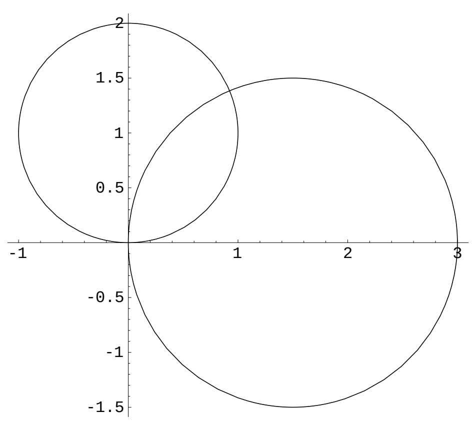
12.



The loop in the first quadrant is described once as θ varies from 0 to $\pi/3$. So the required area is

$$A = \frac{3}{2}\int_0^{\pi/3} \sin^2 3\theta \, d\theta = \frac{3}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta = \frac{\pi}{4}.$$

32.

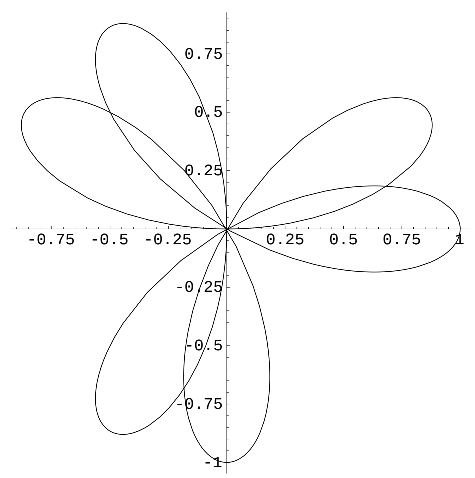


The intersection, in the first quadrant, corresponds to $\theta = \tan^{-1} b/a$. Notice that the area between the two curves is given by the **sum**

$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{\tan^{-1} b/a} a^2 \sin^2\theta \, d\theta + \frac{1}{2}\int_{\tan^{-1} b/a}^{\pi/2} b^2 \cos^2\theta \, d\theta \\
&= \frac{a^2}{4}\int_0^{\tan^{-1} b/a} (1-\cos 2\theta)\, d\theta + \frac{b^2}{4}\int_{\tan^{-1} b/a}^{\pi/2} (1+\cos 2\theta)\, d\theta \\
&= \frac{a^2}{4}\left[\tan^{-1}\frac{b}{a} - \frac{ab}{a^2+b^2}\right] + \frac{b^2}{4}\left[\frac{\pi}{2} - \tan^{-1}\frac{b}{a} - \frac{ab}{a^2+b^2}\right] \\
&= \frac{a^2-b^2}{4}\tan^{-1}\frac{b}{a} + \frac{\pi b^2}{8} - \frac{ab}{4}.
\end{aligned}
$$

It's clear from the original problem that this answer should be symmetric in a and b; you should make use of various trigonometric identities to check that is indeed the case.

38.



The intersections occur when $\tan 3\theta = 1$. Each curve is described once as θ varies from 0 to π and so we seek the values of θ in $[0,\pi]$ with $\tan 3\theta = 1$. These are $\theta = \pi/12,\ 5\pi/12,\ 3\pi/4$. These values show that points of intersection have polar coordinates $(1/\sqrt{2}, \pi/12),\ (-1/\sqrt{2}, 5\pi/12),\ (1/\sqrt{2}, 3\pi/4)$. In addition, the origin is a point of intersection.

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4. We first assume the directrix is $x = 4$ and the eccentricity $e = 1/2$. This yields the polar equation $r = 2/(1+(\cos\theta)/2) = 4/(2+\cos\theta)$. The required ellipse is obtained by rotating this one clockwise about the origin through $\pi/2$, so its equation is

$$r = \frac{4}{2+\cos(\theta+\pi/2)} = \frac{4}{2-\sin\theta}.$$

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6. The directrix is the horizontal line $y = 2$. First we assume the directrix is $x = 2$, the ellipse would then have polar equation $r = 6/(5+3\cos\theta)$. So the required ellipse has polar equation

$$r = \frac{6}{5+3\cos(\theta-\pi/2)} = \frac{6}{5+3\sin\theta}.$$

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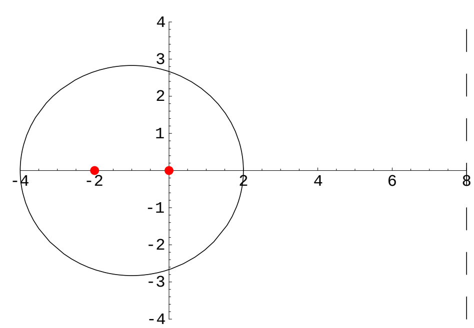
16. The equation can be rewritten as

$$r = \frac{8/3}{1+(1/3)\cos\theta}$$

from which it is clear that the eccentricity is $1/3$ and therefore this is the equation of an ellipse. We also note that the directrix has equation $x = 8$ (because $d = 8$).



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24. For the first curve, we have

$$\frac{dr}{d\theta} = \frac{c\sin\theta}{(1+\cos\theta)^2} = \frac{r\sin\theta}{1+\cos\theta},$$

and for the second

$$\frac{dr}{d\theta} = -\frac{r\sin\theta}{1-\cos\theta}.$$

The slope of the tangent to the first curve is given by

$$\frac{dy}{dx} = \frac{r\cos\theta + (r\sin^2\theta)/(1+\cos\theta)}{-r\sin\theta + (r\sin\theta\cos\theta)/(1+\cos\theta)} = -\frac{1+\cos\theta}{\sin\theta}.$$

A similar calculation for the second curve yields

$$\frac{dy}{dx} = \frac{1-\cos\theta}{\sin\theta}.$$

The product of these slopes is -1 and so the curves are orthogonal at any point of intersection.