CALCULUS III – ANSWERS FALL 1999
TEST 2

In order to get full credit, all answers must be accompanied by appropriate justifications.

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1. Find the limits of the sequences whose n-th terms are as follows:

a) $a_n = \dfrac{n^2-1}{2n^2+10n+3}$; **b)** $a_n = \left(\dfrac{n+5}{n}\right)^n$; **c)** $a_n = (2^n+3^n)^{1/n}$;

d) $a_n = \left(1-\dfrac{1}{n^2}\right)^n$.

(15 points)

. . . .

a)

$$a_n = \frac{1-1/n^2}{2+10/n+3/n^2} \to \frac{1}{2};$$

b)

$$a_n = \left(1+\frac{5}{n}\right)^n \to e^5;$$

c)

$$a_n = 3\left(1+\left(\frac{2}{3}\right)^n\right)^{1/n} \to 3;$$

d)

$$a_n = \left(1-\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right)^n \to \frac{1}{e}e = 1.$$

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2. Find the n-th partial sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}.$$

Use this to find the sum of the series.

. . . .

$$\begin{aligned} s_n &= \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) \\ &= 1 - \frac{1}{n+1} \to 1. \end{aligned}$$

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3. Use the integral test to estimate the sum of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^2},$$

accurate to one decimal place.

. . . .

Put $f(x) = 1/x^2$ and note that f is positive and decreasing on $[1, \infty)$. Consequently,

$$R_n = \sum_{k=n+1}^{\infty} \frac{1}{k^2} \leqslant \int_n^{\infty} f(x)\, dx = \frac{1}{n}.$$

Therefore, if $s = \sum_{k=1}^{\infty} 1/k^2$, we have

$$\sum_{k=1} n \frac{1}{k^2} \leqslant s \leqslant \frac{1}{n} + \sum_{k=1} n \frac{1}{k^2}.$$

Using a calculator we get $1.558 \leqslant s \leqslant (1/11) + 1.558 = 1.649$. So, to one decimal place, we have $s = 1.6$.

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4. Determine whether or not each of the following series converges:

$$\textbf{a)}\quad \sum_{k=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^k; \qquad \textbf{b)}\quad \sum_{k=1}^{\infty}\left(1-\frac{1}{k}\right)^k; \qquad \textbf{c)}\quad \sum_{k=1}^{\infty}\frac{1}{k^2+2k+2}.$$

(15 points)

. . . .

a) This is a geoemtric series with common ratio $1/\sqrt{2} < 1$ and is therefore convergent.

b) Notice that $(1-1/k)^{-1} \to e^{-1} \neq 0$ as $k \to \infty$, and so the series diverges.

c) Put $a_k = 1/(k^2+2k+2)$ and $b_k = 1/k^2$ then $a_k/b_k \to 1$ as $k \to \infty$. Thus $\sum a_k$ and $\sum b_k$ either both converge or both diverge. The latter converges and so our series converges.

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5. Determine whether or not each of the following series converges distinguishing between absolute and conditional convergence:

$$\textbf{a)}\quad \sum_{k=1}^{\infty}\frac{(-1)^k \sin k}{k^2}; \qquad \textbf{b)}\quad \sum_{k=1}^{\infty}\frac{(-1)^k}{k}; \qquad \textbf{c)}\quad \sum_{k=1}^{\infty}\frac{(k!)^2}{(2k)!}.$$

(20 points)

. . . .

a) Note that $|(-1)^k \sin k/k^2| \leqslant |1/k^2|$ and so the comparison test shows that the series converges absolutely.

b) Note that $|(-1)k/k| = 1/k$ and $\sum 1/k$ diverges. We may use the alternating series test since $1/k > 1/(k+1) \to 0$. Consequently the series converges conditionally.

c) Using the ratio test, we have

$$\frac{((k+1)!)^2}{(2k+2)!}\frac{(2k)!}{(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} \to \frac{1}{4} < 1.$$

It follows that the series converges (absolutely).

6. Find the interval of convergence of each of the following power series:

$$\textbf{a)}\quad \sum_{k=0}^{\infty} \frac{k^2 x^k}{10^k}; \qquad \textbf{b)}\quad \sum_{k=1}^{\infty} \frac{(x-1)^k}{\sqrt{k}}.$$

(20 points)

. . . .

a) The ratio test gives

$$\frac{(k+1)^2|x|^{k+1}}{10^{k+1}}\frac{10^k}{k^2|x|^k} = \left(\frac{k+1}{k}\right)^2 \frac{|x|}{10} \to \frac{|x|}{10}.$$

So the series converges absolutely in $(-10, 10)$ and diverges when $|x| > 10$. When $x = \pm 10$ the series is $\sum_{k=0}^{\infty} k^2$ or $\sum_{k=0}^{\infty}(-1)^k k^2$. These both diverge since the terms of the series do not approach zero. So the interval of convergence is $(-10, 10)$.

b) The ratio test gives

$$\frac{|x-1|^{k+1}}{\sqrt{k+1}}\frac{\sqrt{k}}{|x-1|^k} = |x-1|\sqrt{\frac{k}{k+1}} \to |x-1|.$$

Consequently the series converges absolutely in $(0, 2)$ and diverges when $|x-1| > 1$. When $x = 0$ the series is $\sum(-1)^k/\sqrt{k}$ which converges, by the alternating series test. When $x = 2$ the series is $\sum 1/\sqrt{k}$ which diverges. So the interval of converegnce is $[0, 2)$.