CALCULUS III FALL 1999

TEST 3

In order to get full credit, all answers must be accompanied by appropriate justifications.

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1. Find the Taylor series for $f(x) = \sin x$ at $a = \pi/6$.

(15 points)

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Note that

$$f^{(4n)}\left(\frac{\pi}{6}\right) = \frac{1}{2}; \qquad f^{(4n+1)}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2};$$
$$f^{(4n+2)}\left(\frac{\pi}{6}\right) = -\frac{1}{2}; \qquad f^{(4n+3)}\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}.$$

Consequently, the Taylor series at $a = \pi/6$ is

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}.$$

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2. Use the binomial series to find the Maclaurin series for

$$f(x) = \frac{1}{\sqrt[4]{16-x}},$$

and give the radius of convergence of this series.

(15 points)

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Note that $f(x) = (1/2)(1 - x/16)^{-1/4}$ and so, using the binomial series, we have

$$f(x) = \frac{1}{2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1/4)(-5/4)\cdots(-1/4-n+1)}{n!}\left(-\frac{x}{16}\right)^n\right]$$
$$= \frac{1}{2}\left[1 + \sum_{n=0}^{\infty}\frac{1.5\ldots(4n-3)}{4^{3n}n!}x^n\right].$$

This converges when $|x/16| < 1$ and so the radius of convergence is 16.

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3. How many terms of the Maclaurin series for $\ln(1+x)$ do you need to estimate $\ln 1.4$ to within 0.001?

(20 points)

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The Maclaurin series for $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad \text{for } -1 < x \leqslant 1.$$

Putting $x = 0.4$ gives $\ln(1.4) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \frac{4^n}{10^n}$. This series satisfies the alternating series test. So if

$$R_N = \ln(1.4) - \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} \frac{4^n}{10^n} \qquad \text{then } |R_N| \leqslant \frac{1}{N+1} \frac{4^{N+1}}{10^{N+1}}.$$

Consequently, we want to find N so that $(N+1)10^{N-2} > 4^{N+1}$. Your friendly calculator shows that this is true for the first time when $N = 5$. Consequently we get the desired accuracy from the first five terms of the series.

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4. Line segments are drawn from the vertex of a parallelogram to the midpoints of the opposite sides. Show that they trisect a diagonal.

(20 points)

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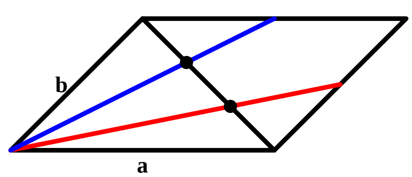

Let **a**, **b** be vectors representing two sides of the parallelogram. The midpoints of two of the sides are $\mathbf{a} + \frac{1}{2}\mathbf{b}$ and $\mathbf{b} + \frac{1}{2}\mathbf{a}$. Consequently the points of intersection with the diagonal are determined by

$$t\left(\mathbf{b} + \frac{1}{2}\mathbf{a}\right) = \mathbf{b} + \alpha(\mathbf{a} - \mathbf{b}) \qquad \text{and} \qquad s\left(\mathbf{a} + \frac{1}{2}\mathbf{b}\right) = \mathbf{b} + \beta(\mathbf{a} - \mathbf{b}).$$

This gives $\alpha = 1/3$, $\beta = 2/3$. So the points of intersection with the diagonal are one third and two thirds of the way from **b** to **a**.

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5. Given the points $A(1,0,1)$, $B(2,3,0)$, $C(-1,1,4)$ and $D(0,3,2)$, find the volume of the parallelepiped with adjacent edges AB, AC, AD.

(10 points)

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The listed edges are given by the vectors $\mathbf{a} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{b} = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, $\mathbf{c} = -\mathbf{i} + 3\mathbf{j} + \mathbf{k}$. The required volume is therefore $V = |\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$. We have $\mathbf{b} \times \mathbf{c} = -8\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ and so $V = -8 - 3 + 5| = 6$.

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6. Find an equation of the plane passing through the points $(-1,2,0)$, $(2,0,1)$ and $(-5,3,1)$.

(10 points)

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The vectors $\mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{b} = -4\mathbf{i} + \mathbf{j} + \mathbf{k}$ are parallel to the plane. So $\mathbf{n} = \mathbf{a} \times \mathbf{b}$ is normal to the plane. Note that $\mathbf{n} = -3\mathbf{i} - 7\mathbf{j} - 5\mathbf{k}$. So the required equation is $-3x - 7y - 5z = -11$ or $3x + 7y + 5z = 11$.

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7. Find the curvature of the graph of $y = x^4$ at the point $(1,1)$.

(20 points)

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The vector equation of the graph is $\mathbf{r}(x) = x\mathbf{i} + x^4\mathbf{j}$. Hence $\mathbf{r}'(x) = \mathbf{i} + 4x^3\mathbf{j}$ and $\mathbf{r}''(x) = 12x^2\mathbf{j}$. At $x = 1$ we have $\mathbf{r}'(1) = \mathbf{i} + 4\mathbf{j}$ and $\mathbf{r}''(1) = 12\mathbf{j}$. So $\mathbf{r}'(1) \times \mathbf{r}''(1) = 12\mathbf{k}$. Consequently, the require curvature is

$$\kappa = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{12}{17^{3/2}}.$$