CALCULUS III FALL 1999

REVIEW 3

Sic Hoc Legere Scis Nimium Eruditionis Habes

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1. For each of the following functions find the Taylor polynomial in $(x-a)$ of degree n for the given values of a and n:
a). $f(x) = \sqrt[3]{x+8}$, $a = 0,\ n = 3$; b). $f(x) = \sqrt[3]{x}$, $a = 125,\ n = 3$;
c). $f(x) = \tan^{-1} x$, $a = 1,\ n = 3$.

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2. Use Taylor's Theorem to calculate $\sqrt[3]{120}$ accurate to three decimal places.

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3. Show that the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!}$$

can be used to calculate $\sin x$ for all values of x that correspond to angles between $0°$ and $90°$.

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4. Use power series to estimate $\int_0^1 \ln(1+x^2)\,dx$ accurate to 1 decimal place.

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5. Find a power series that represents $1/(1+x)^3$ on the interval $(-1, 1)$.

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6. Find the Maclaurin series for $\sin^2 x$.

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7. Find the first five terms of the Taylor series for e^x about the point $x = 2$.

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8. Use the simplest method you can think of to find the first three nonzero terms of the Maclaurin series for each of the following:

$$\frac{1}{1-x^3} \qquad \sqrt{1+x^2} \qquad x\sec x$$

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9. If $f(x) = (\sin x)/x$ for $x \neq 0$ and $f(0) = 1$, find $f^{(k)}(0)$.

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10. If $\mathbf{u}+\mathbf{v}$ is perpendicular to $\mathbf{u}-\mathbf{v}$, what can you say about the relative magnitudes of $\mathbf{u}$ and $\mathbf{v}$?

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11. Let the sides of a regular hexagon be drawn as vectors with the head of each at the tail of the next.

a) If $\mathbf{a}$ and $\mathbf{b}$ are vectors represented by consecutive sides, find the other four vectors in terms of $\mathbf{a}$ and $\mathbf{b}$.
b) What is the vector sum of all six vectors?

12.

a) Derive a formula for a vector that bisects the angle between two vectors $\mathbf{a}$ and $\mathbf{b}$.
b) Line segments are drawn from the vertex of a parallelogram to the midpoints of opposite sides. Show that they trisect a diagonal.

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13. Find the angle between a). $2\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $3\mathbf{i}-4\mathbf{k}$; and b). the x-axis and $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

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14. Use vector methods to show that, if PQR is a triangle, there is a triangle whose sides are parallel and equal in lengths to the medians of PQR.

15. Find the area of the triangle with vertices $P(1,3,-2)$, $Q(1,0,1)$ and $R(-3,-2,2)$. Find the volume of the parallelepiped with adjacent edges OP, OQ, and OR.

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16. Find the equations of the line a). perpendicular to the yz-plane, passing through $(1,2,3)$; and b). passing through the origin and parallel to the line

$$x-3=\frac{y+2}{4}=1-z.$$

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17. Use vector methods to:

a) Prove the parallelogram equality, that is, the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its sides.
b) Prove that the angle subtended at the circumference by a diameter of a circle is a right angle.

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18. Find the distance between a). the planes $x+2y+3z=5$ and $x+2y+3z=19$; and b). the origin and the plane through $(1,2,3)$, $(0,-1,1)$ and $(2,0,2)$.

19.

a) Find a unit vector perpendicular to both $3\mathbf{i}+\mathbf{j}$ and $2\mathbf{i}-\mathbf{j}-5\mathbf{k}$.
b) Compute $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$, given that $\mathbf{a}=2\mathbf{i}+2\mathbf{j}$, $\mathbf{b}=3\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{c}=8\mathbf{i}$.
c) Express $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ as the sum of a vector parallel, plus a vector orthogonal to $2\mathbf{i}+4\mathbf{j}-2\mathbf{k}$.

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20. Find the volume of the parallelepiped with edges AB, AC and AD where $A=(3,2,1)$, $B=(4,2,1)$, $C=(0,1,4)$ and $D=(0,0,7)$.

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21. Find an equation of the plane passing through the origin parallel to the vectors $\mathbf{a}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+5\mathbf{k}$.

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22. Show that the vectors $\mathbf{i}-\mathbf{j}$, $\mathbf{j}-\mathbf{k}$ and $\mathbf{k}-\mathbf{i}$ are parallel to a plane. Find the equation of a plane passing through $(1, 2, 3)$ and parallel to these three vectors.

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23. Find the points on the curve $\mathbf{r}(t) = t\mathbf{i} + (1 - t^2)\mathbf{j}$ at which $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are (a) perpendicular; (b) have the same direction; (c) have opposite directions.

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24. Suppose that $\mathbf{r}'(t)$ and $\mathbf{r}(t)$ are parallel for all t and that $\mathbf{r}'(t)$ is never $\mathbf{0}$. Show that the tangent line at each point passes through the origin.

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25. Find the points at which the curves

$$\mathbf{r}_1(t) = e^t\mathbf{i} + 2\sin\Big(t + \frac{\pi}{2}\Big)\mathbf{j} + (t^2 - 2)\mathbf{k}; \qquad \mathbf{r}_2(u) = u\mathbf{i} + 2\mathbf{j} + (u^2 - 3)\mathbf{k}$$

intersect and find the angle of intersection at each such point.

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26. Find the length of the curve

$$\mathbf{r}(t) = t^2\mathbf{i} + (t^2 - 2)\mathbf{j} + (1 - t^2)\mathbf{k} \qquad 0 \leqslant t \leqslant 2.$$

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27. Let C_1 be the curve

$$\mathbf{r}(t) = (t - \ln t)\mathbf{i} + (t + \ln t)\mathbf{j} \qquad 1 \leqslant t \leqslant e$$

and let C_2 be the graph of $y = e^x$ on the interval $[0, 1]$. Find a relationship between the length of C_1 and the length of C_2.

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28. Find the curvature at the vertices of the hyperbola $xy = 1$.

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29. Find the curvature at the highest point of an arch of the cycloid

$$x(t) = r(t - \sin t), \qquad y(t) = r(1 - \cos t).$$

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30. Show that the curvature of a polar curve $r = f(\theta)$ is given by

$$\kappa = \frac{|[f(\theta)]^2 + 2[f'(\theta)]^2 - f(\theta)f''(\theta)|}{([f(\theta)]^2 + [f'(\theta)]^2)^{3/2}}.$$

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31. Find the curvature of the spiral of Archimedes $r = a\theta$.