Outer Space and its Bordification
I. Outer Space

\textit{Obs:} Connected graph are classifying spaces for their fundamental groups. In particular

\[ \left[ (\Gamma, \ast); (\Delta, \ast) \right] = \text{Hom} \left( \pi_1(\Gamma, \ast); \pi_1(\Delta, \ast) \right) \]

Thus: \( \text{Aut}(F_n) = \{ [\varphi] \mid \varphi : (R_n, \ast) \rightarrow (R_n, \ast), \text{ homotopy equiv.} \} \)

Also: \( \text{Out}(F_n) = \{ [\varphi] \mid \varphi : R_n \rightarrow R_n, \text{ homotopy equiv.} \} \)

\[ R_n = \text{rose with } n \text{ petals} \]
Def: A marking on a graph $\Gamma$ is a homotopy equivalence $g: R_n \rightarrow \Gamma$. $R_n \xrightarrow{\phi} R_n \xrightarrow{g} \Gamma$

Note: Out $F_n = [R_n: R_n]$ acts on markings by precomposition.

A metric on a graph $\Gamma$ assigns a length $>0$ to each edge. $\mu(\Gamma) = \sum_{e: \text{edge}} \mu(e)$

Outer space $X_n$ consists of marked metric graphs $(\Gamma, g, \mu)$ of total volume 1 up to isometry. Reduced: no vertices of deg 1 or 2.

$R_n \xleftarrow{\sim} \xrightarrow{\text{isometry}} [g_1] \rightarrow \Gamma_1$

$R_n \xleftarrow[\sim]{} \xrightarrow{\text{isometry}} [g_2] \rightarrow \Gamma_2$
On each marked graph $(\Gamma, g)$, we can visualize the volume metrics on $\Gamma$ as an open simplex $\sigma$ (edges of $\Gamma$ are the vertices of $\sigma$, their lengths at a point $p \in \sigma \subset X_m$ are the barycentric coordinates).

$$R_2 = \infty$$

triangle not in reduced outer space.
The Spine

Def: Let $\mathcal{C}_n$ be the category of marked graphs of rank $n$ without deg 1 and 2 vertices and without separating edges where morphisms are given by collapsing a forest.

$K_n$ is the geometric realization of $\mathcal{C}_n$, i.e.:

$K_n$ is a $\Delta$-complex

Fact: $K_n$ is a simplicial complex.

vertices: objects
simplices: chains of morphisms
Rem: $K_n$ can be realized as an equivariant strong deformation retract inside $X_n$. The vertex $(\Gamma, y)$ corresponds to the barycenter of $\sigma(\Gamma, y)$.

You define a flow on each open simplex. They fit together.
Obs: $\dim X_n = 3n - 4$
\[\dim K_n = 2n - 3\]

'Euler characteristic (minimum vertex deg $\geq 3$!)

Cor: The action of $\text{Out}(F_n)$ on $K_n$ is compact.

'There are only finitely many graphs of rank $n$ without deg 1 or 2 vertices.
II Connectivity at infinity

Def: A loc. compact space $X$ is $n$-connected @ $\infty$ if for every compact $C \subseteq X$ there is a compact $D \supseteq C$ such that any cont. map $f : S^m \to X \setminus D$ $(m < n)$ extends to a cont. map $\bar{f} : B^{m+1} \to X \setminus C$.

Ex: $(-1)$-connected @ $\infty \iff$ non-compact

$0$-connected @ $\infty \iff$ has 1 end.
Rem: Let $G$ be a group of type $F_n$ and let $X$ and $Y$ be two classifying CW complexes for $G$ with finite $n$-skeleton.

Then $X$ is $n$-connected $\Rightarrow Y$ is $n$-connected $\Rightarrow$

Thus: Being $n$-connected $\Rightarrow$ is an invariant of the group $G$. 
Fact: $K_n$ is contractible   \[ \Rightarrow \text{Out}(F_n) \text{ is of type } F_0 \]

Cell stabilizers are finite

Fact: $\text{Out}(F_n)$ is virtually torsion free. In fact, the kernel $O_n$ of the composition

$$\text{Out}(F_n) \to \text{GL}_n(\mathbb{Z}) \to \text{GL}_n(\mathbb{Z}/3\mathbb{Z})$$

has no torsion.

Thus: $\text{Out}(F_n)$ has a well-defined virtual connectivity @ $\infty$, namely the connectivity $\@\infty$ of $K_n$. 
Def: Suppose a group $G$ acts freely, cocompactly by cell permuting homeos on a contractible CW complex $X$. $G$ is a duality group of dimension $n$ if $H^i(G; \mathbb{Z}G) = 0$ for $i \neq n$ and $H^n(G; \mathbb{Z}G)$ is torsion-free.

Note: $H^i(G; \mathbb{Z}G) = H^i_c(X; \mathbb{Z})$.

Rem: $H^i_c(K_n; \mathbb{Z}) = 0$ for $i > 2n - 3$ (dim $K_n$)

Geoghegan–Mihalik: If $G$ and $X$ are as above and $X$ is $k$-conn. @ $\infty$, then $H^i_c(X; \mathbb{Z}) = 0$ for $i \leq k+1$. 
Thm (Bestvina–Feighn):

1. Bordified outer space is $(2n-5)$-connected at $\infty$.
2. $\text{Out}(F_n)$ is virtually a duality group of dimension $2n-3$.

Rem.: To infer (2) from (1), one needs to show that $H_*^{2n-3}(X; \mathbb{Z})$ is torsion-free.

Rem.: Let $R_n$ be the nerve of the cover of $K_n$ by stars of roses. It suffices to understand $R_n$. 

Ⅲ The Bordification

Rem: Simplices $\sigma (\Gamma , g)$ are missing faces:

\[ \Gamma = \bigcirc \quad \Gamma = \bigtriangleup \qquad \Gamma = \bigcirc\bigtriangleup\]

\[\text{back faces are missing}\]

Idea: Define jewels $J(\Gamma , g)$ by "shaving off" faces at $\infty$. 
Warning: one needs to be careful: \( \emptyset = \emptyset \)

\[
\begin{array}{c}
\text{not good (losing faces)}
\end{array}
\]

\[
\begin{array}{c}
\text{good}
\end{array}
\]

shave off \( \varepsilon \dim (\text{face}) + 1 \) \( \rightarrow \) all faces remain
Def: Jewel space is obtained by gluing jewels $J(\sigma, \phi)$ according to the same pattern as the simplices $\sigma(\sigma, \phi)$ are glued together in outer space.

$\Rightarrow J_n$ is an equivariant, cocompact deformation retract of outer space.
Fact: Each vertex of $J_n$ lies in a jewel $J(R,q)$ where the graph is a rose.

The jewel $J(R,q)$ of a rose is a permutohedron. The vertices correspond to orderings of the petals.

The jewel $J(P,q)$ is the convex hull of its vertices / its rose faces.

$\Gamma = \infty$

back faces are at $\infty$. 
Complexes:

\( J_n \): jewel complex

\( S_n \): simplicial core of \( J_n \) obtained by gluing simplices \( S(\Gamma, g) = \{ \text{vertices of } J(\Gamma, g) \} = \{ \text{ordered roses collapsed from } (\Gamma, g) \} \)

\( J_n \approx S_n \) by a nerve-cover argument

\( R_n \): rose complex obtained by gluing simplices \( R(\Gamma, g) = \{ \text{roses collapsed from } (\Gamma, g) \} \)

\( S_n \rightarrow R_n \), forget order has contractible fibers \( \approx \) hom. equiv.
Thm: $R_n$ is $(2n-5)$-connected. Rose complex: nerve of cover by rose stays

Strategy: we order the marked roses $S_1, S_2, S_3, \ldots$ s.t.

\[ R_{\geq t} := \text{subcomplex spanned by} \{ S_t, S_{t+1}, \ldots \} \]

$R_{\geq t}$ is obtained from $R_{\geq t+1}$ by coming off $1k^\uparrow (S_t)$. 

Claim: $R_{\geq t}$ is $(2n-5)$-connected for each $t$.

Proof: $t = 0: R_{\geq 0}$ is contractible

$t \geq 1: 1k^\uparrow (S_t)$ is $(2n-4)$-spherical.

Note: The claim implies that $R_n$ is $(2n-5)$-connected at $\infty$. 
Rem: \( H^i(C_{R_{20}}, R_{2t}) \cong \tilde{H}^{i-1}(R_{2t}) \)

Thus: \( H^i_c(R_{20}) = \lim_m \tilde{H}^{i+m}(R_{2t}) \) \text{ direct limit}

Claim: \( \tilde{H}^j(R_{2t}) = \begin{cases} 0 & : j \neq 2n-4 \\ \bigoplus_{s \leq t} \tilde{H}^{2n-4}(\text{lk}^+(s_s)) & : j = 2n-4 \end{cases} \)

Induct on \( t \): \( R_{2t} = R_{2t+1} \cup \text{lk}^+(s_t) \text{ Cone (lk}^+(s_t)) \)

\[ 0 \to \tilde{H}^{2n-4}(R_{2t}) \to \tilde{H}^{2n-4}(R_{2t+1}) \to \tilde{H}^{2n-4}(\text{lk}^+) \to 0 \]

and the sequence splits.
Some details on the order:

$S$ = marked rose

$w$ = conj. class in $F_n$

$|S|_w$ = length of $w$ in $S$ (pulled tight)

$W_0 = \{ x_i, x_i x_i^\pm \}$  $W = \{ w_1, w_2, \ldots \text{ all conj. classes} \}$

$|S|_0 := \sum_{w \in W_0} |S|_w$

$h(S) := (|S|_0, |S|_{w_1}, |S|_{w_2}, \ldots)$  height

(lexicographic order)
Facts: 1) $\mathcal{M}$ marked roses $\mathcal{F}$ is well-ordered by height.

$\Rightarrow$ 1a) Sublevel sets are cocompact.

1b) no horizontal edges.

Where is the work?

prove that ascending links $lk^i(s)$ are $(2n-5)$-connected.
Ⅵ Ideal Edges

Ideal edges describe one-edge blow ups

\[ \xrightarrow{\text{\textbullet}} \]
Def: An ideal edge is a partition of the set of half-edges of a marked rose into two sets (sides).

Rules: 1) each side has at least two elements
2) the partition splits an edge.

Forbidden:
- deg 2
- deg 1
- separating edge
Blow up trees correspond to sets of pairwise compatible ideal edges

\[ \text{Def: } I(s) : \text{key complex of pairwise comp. ideal edges at } s. \]
Key: $I^k(s) \sim I^k(s)$

Claim: $I^k(s)$ is $(2m-4)$-spherical.

The proof is by induction and needs to show a more general statement about partitions of

\[ \{ x_1, x_1', \ldots, x_m, x_m', y_1, \ldots, y_k \} \]

\[ \Rightarrow (2m+k-4) \text{-spherical, provided } 2m+k-4 \geq 0. \]