Outer Space and its Bordification

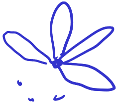
I Outer Space

Obs: Connected graph are classifying spaces for their fundamental groups. In particular

$$[(\Gamma,\bullet);(\Delta,*)] = \mathrm{Hom}(\pi_1(\Gamma,\bullet);\pi_1(\Delta,*))$$

Thus: $\mathrm{Aut}(F_n) = \{[\varphi] \mid \varphi:(R_n,\bullet)\to(R_n,\bullet),\ \text{homotopy equiv.}\}$

Also: $\mathrm{Out}(F_n) = \{[\varphi] \mid \varphi: R_n \to R_n,\ \text{homotopy equiv.}\}$

$R_n =$  rose with n petals

Def: A <u>marking</u> on a graph Γ is a homotopy equivalence $g: R_n \to \Gamma$.

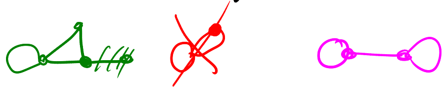
$$R_n \xrightarrow{\varphi} R_n \xrightarrow{g} \Gamma$$

<u>Note</u>: $\mathrm{Out}\, F_n = [R_n; R_n]$ acts on markings by precomposition.

A <u>metric</u> on a graph Γ assigns a length >0 to each edge.

$$\mu(\Gamma) = \sum_{e:\text{edge}} \mu(e)$$

F_2



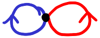
Outer space X_n consists of marked metric graphs (Γ, g, μ) of total volume 1 up to isometry.

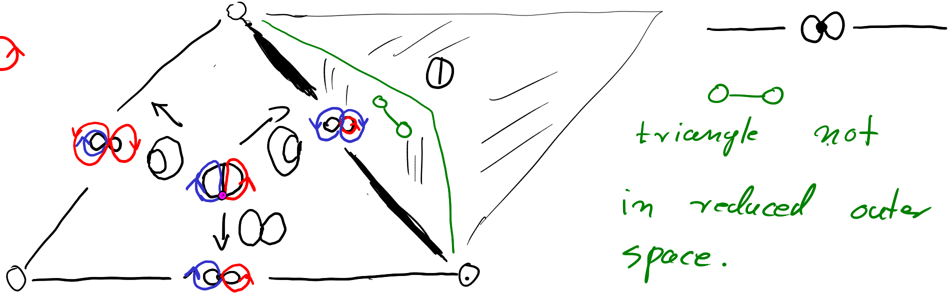
no vertices of deg 1 or 2.

<u>reduced</u>: no sep. edges

$$\begin{tikzcd}
 & \Gamma_1 \arrow[dd, "\text{isometry}"] \\
R_n \arrow[ur, "{[g_1]}"] \arrow[dr, "{[g_2]}"'] \arrow[r, phantom, "\sim"] & \\
 & \Gamma_2
\end{tikzcd}$$

Picture: On each marked graph (Γ, g), we can visualize the vol. 1 metrics on Γ as an open simplex σ (edges of Γ are the vertices of σ, their lengths at a point $p \in \sigma \subseteq X_n$ are the barycentric coordinates).

$R_2 =$ 



triangle not in reduced outer space.

The Spine

Def: Let $\mathcal{C}_n$ be the category of marked graphs of rank n without deg 1 and 2 vertices and without seperating edges where morphisms are given by collapsing a forest.

K_n is the geometric realization of $\mathcal{C}_n$, i.e.:

K_n is a Δ-complex **Fact:** K_n is a simplicial cx.

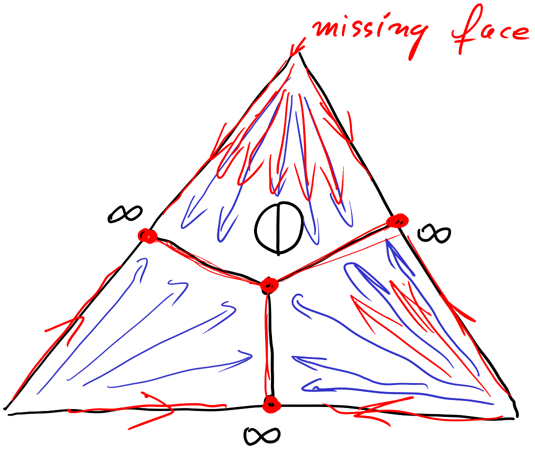
vertices : objects

simplices : chains of morphisms

$$\begin{tikzcd} \Gamma_1 \arrow[rr] \arrow[dr] & & \Gamma_3 \\ & \Gamma_2 \arrow[ur] & \end{tikzcd}$$

triangle

Rem: K_n can be realized as an equivariant strong deformation retract inside X_n. The vertex (Γ, g) corresponds to the barycenter of $\sigma(\Gamma, g)$.



⌈ You define a flow on each open simplex. They fit together. ⌋

<u>Obs</u>: $\dim X_n = 3n-4$

$\dim K_n = 2n-3$

⌜Euler characteristic (minimum vertex deg ≥ 3!)⌟

<u>Cor</u>: The action of $Out(F_n)$ on K_n is cocompact.

⌜There are only finitely many graphs of rank n without deg 1 or 2 vertices.⌟

II Connectivity at infinity

<u>Def</u>: A loc. compact space X is <u>n-connected @∞</u> if for every compact $C \subseteq X$ there is a compact $D \supseteq C$ such that any cont. map $f : S^m \to X \setminus D$ $(m \leq n)$ extends to a cont. map $\bar{f} : \mathbb{B}^{m+1} \to X \setminus C$.

<u>Ex</u>: (-1)-connected @∞ $\iff$ non-compact

0-connected @∞ $\iff$ has 1 end.

Rem: Let G be a group of type F_n and let X and Y be two classifying CW complexes for G with finite n-skeleton.

Then $\tilde{X}$ is n-connected@∞ $\Leftrightarrow$ $\tilde{Y}$ is n-connected@∞

Thus: Being n-connected@∞ is an invariant of the group G.

$$\left.\begin{array}{l}\underline{\text{Fact}}: \quad K_n \text{ is contractible} \\ \text{Cell stabilizers are finite}\end{array}\right\} \underset{\text{Brown}}{\Longrightarrow} \quad \text{Out}(F_n) \text{ is of type } F_\infty$$

<u>Fact</u>: Out(F_n) is virtually torsion free. In fact, the kernel O_n of the composition

$$\text{Out}(F_n) \to GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/3\mathbb{Z})$$

has no torsion.

<u>Thus</u>: Out(F_n) has a well-defined virtual connectivity@∞, namely the connectivity@∞ of K_n.

Def: Suppose a group G acts freely, cocompacty by cell permuting homeos on a contractible CW complex X. G is a duality group of dimension n if $H^i(G;\mathbb{Z}G)=0$ for $i\neq n$ and $H^n(G;\mathbb{Z}G)$ is torsion-free.

Note: $H^i(G;\mathbb{Z}G) = H^i_c(X;\mathbb{Z})$.

Rem: $H^i_c(K_n;\mathbb{Z})=0$ for $i>2n-3$ ($\dim K_n$)

Geoghegan - Mihalík: If G and X are as above and X is k-conn.@ ∞, then $H^i_c(X;\mathbb{Z})=0$ for $i\leq k+1$.

Thm (Bestvina - Feighn):

(1) Bordified outer space K_n is $(2n-5)$-connected @ ∞.

(2) $Out(F_n)$ is virtually a duality group of dimension $2n-3$.

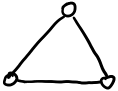
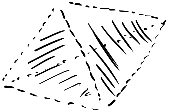
Rem: To infer (2) from (1), one needs to show that $H_c^{2n-3}(X;\mathbb{Z})$ is torsion-free.

Rem: Let R_n be the nerve of the cover of K_n by stars of roses. It suffices to understand R_n.

III The Bordification

<u>Rem</u>: Simplices $\sigma(\Gamma, g)$ are missing faces:

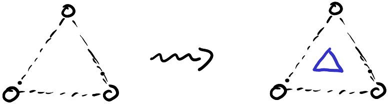
$\Gamma =$  $\Gamma =$  $\Gamma =$ 

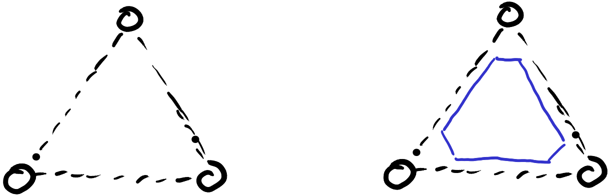
back faces are missing

<u>Idea</u>: Define <u>jewels</u> $J(\Gamma, g)$ by "shaving off" faces at ∞.

__Warning__: one needs to be careful: $\Gamma =$ 



not good (loosing faces)



good

shave off $\varepsilon^{\dim(\text{face})+1}$ $\rightsquigarrow$ all faces remain

Def: Jewel space is obtained by gluing jewels $J(\Gamma, g)$ according to the same pattern as the simplices $\sigma(\Gamma, g)$ are glued together in outer space.

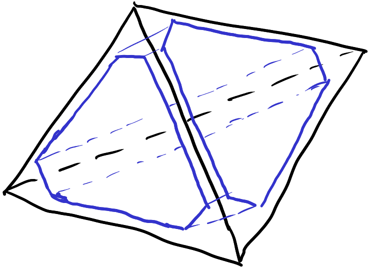
$\rightsquigarrow$ J_n is an equivariant, cocompact deformation retract of outer space.

<u>Fact</u>: Each vertex of $\mathcal{J}_n$ lies in a jewel $\mathcal{J}(R,q)$ where the graph is a rose.

The jewel $\mathcal{J}(R,q)$ of a rose is a permutohedron.

The vertices correspond to orderings of the petals.

The jewel $\mathcal{J}(\Gamma,q)$ is the convex hull of its vertices / its rose faces.



$\Gamma =$ 

back faces are at ∞.

Complexes :

$\mathcal{J}_n$: jewel complex

S_n : simplicial core of $\mathcal{J}_n$ obtained by gluing

simplices $S(\Gamma, g) = \{$ vertices of $\mathcal{J}(\Gamma, g) \}$.
$= \{$ ordered roses collapsed from $(\Gamma, g)\}$

$\mathcal{J}_n \simeq S_n$ by a nerve-cover argument

R_n : rose complex obtained by gluing simplices

$R(\Gamma, g) = \{$ roses collapsed from $(\Gamma, g) \}$

$S_n \twoheadrightarrow R_n$, forget order has contractible fibers $\rightsquigarrow$ hom. equiv.

Bestvina - Feighn, B-Smillie - Vogtmann

Rose complex: nerve of cover by rose stars.

Thm: R_n is $(2n-5)$-connected @ ∞.

Strategy: we order the marked roses $\rho_1, \rho_2, \rho_3, \dots$ s.th.:

$R_{\geq t}$:= subcomplex spanned by $\{\rho_t, \rho_{t+1}, \dots\}$

$R_{\geq t}$ is obtained from $R_{\geq t+1}$ by coning off $lk^{\uparrow}(\rho_t)$. asc. link

claim: $R_{\geq t}$ is $(2n-5)$-connected for each t.

pf: $t=0$: $R_{\geq 0}$ is contractible

$t \geq 1$: $lk^{\uparrow}(\rho_i)$ is $(2n-4)$-spherical.

$\Rightarrow R_{\geq t}$ is $(2n-5)$-conn. for each t.

Note: The claim implies that R_n is $(2n-5)$-connected @ ∞.

Rem: $H^i(R_{\geq 0}, R_{\geq t}) \cong \tilde{H}^{i-1}(R_{\geq t})$

Thus: $H^i_c(R_{\geq 0}) = \lim \tilde{H}^{i-1}(R_{\geq t})$ direct limit

claim: $\tilde{H}^j(R_{\geq t}) = \begin{cases} 0 & : j \neq 2n-4 \\ \bigoplus_{s<t} \tilde{H}^{2n-4}(\mathrm{lk}^\uparrow(\varrho_s)) & : j = 2n-4 \end{cases}$

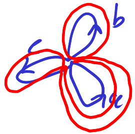
⌜ induct on t: $R_{\geq t} = R_{\geq t+1} \cup_{\mathrm{lk}^\uparrow(\varrho_t)} \mathrm{Cone}(\mathrm{lk}^\uparrow(\varrho_t))$

$$\rightsquigarrow 0 \to \tilde{H}^{2n-4}(R_{\geq t}) \to \tilde{H}^{2n-4}(R_{\geq t+1}) \to \underbrace{\tilde{H}^{2n-4}(\mathrm{lk}^\uparrow)}_{\text{free abelian fin. rank}} \to 0$$

and the sequence splits. ⌟

$a b \bar{a} c$

Some details on the order:



ρ : marked rose

w : conj. class in F_n

$\rightsquigarrow$ $|\rho|_w$: length of w in ρ (pulled tight)

$W_0 = \{ x_i, \; x_i x_j^{\pm} \}$ $\quad$ $W = \{ w_1, w_2, \dots$ all conj. classes$\}$

$$|\rho|_0 := \sum_{w \in W_0} |\rho|_w$$

$$h(\rho) := (|\rho|_0, |\rho|_{w_1}, |\rho|_{w_2}, \dots) \qquad \text{height}$$

(lexicographic order)

Facts: 1) {marked roses} is well-ordered by height.

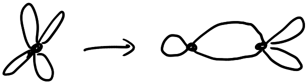
$\Rightarrow$ 1a) Sublevel sets are cocompact.

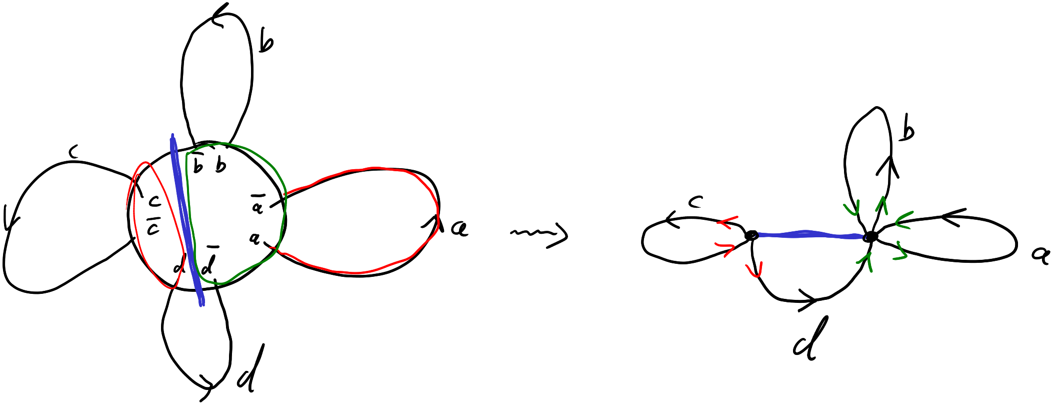
1b) no horizontal edges.

Where is the work?

prove that ascending links $lk^{\uparrow}(s)$ are $(2n-5)$-connected.

VI Ideal Edges

Ideal edges describe one-edge blow ups 

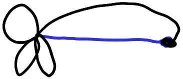


Def: An ideal edge is a partition of the set of half-edges of a marked rose into two sets (*sides*)

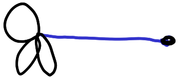
Rules: 1) each side has at least two elements

2) the partition splits an edge

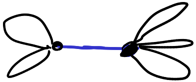
forbidden:



deg 2

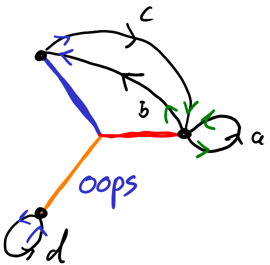


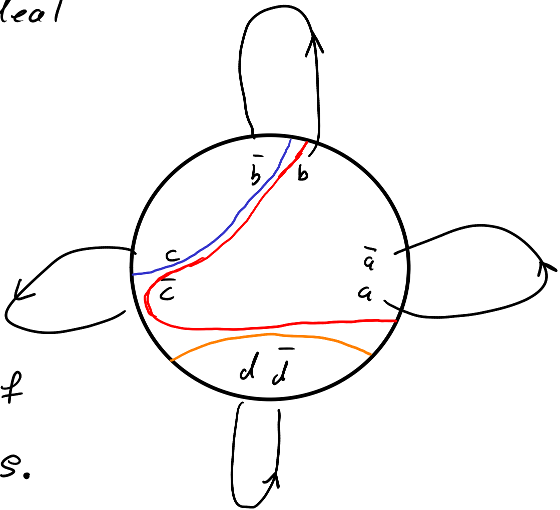
deg 1



separating edge

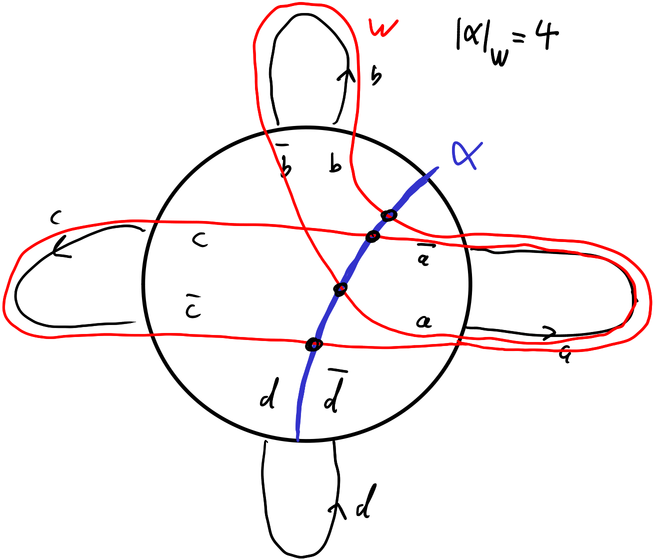
Blow up trees correspond to sets of pairwise compatible ideal edges





<u>Def</u>: $I(S)$: flag complex of pairwise comp. ideal edges at S.

$W = a\,b\,a\,c$



$|\alpha|_w$: # crossings (pulling tight)

$|e|_w$: # traversals

<u>Def</u> : α is <u>ascending</u> if $|\alpha| > |e|$ for some petal e. The complex of ascending ideal edges is $\mathcal{I}^{\uparrow}(\rho)$.

Key : $\mathrm{lk}^{\uparrow}(\varrho) \simeq I^{\uparrow}(\varrho)$

Claim : $I^{\uparrow}(\varrho)$ is $(2n-4)$-spherical.

⌜ The proof is by induction and needs to show a more general statement about partitions of

$$\{x_1, \overline{x_1}, \dots, x_m, \overline{x_m}, y_1, \dots, y_k\}$$

$\rightsquigarrow$ $(2m+k-4)$-spherical, provided $2m+k-4 \geq 0$. ⌟