The dimension of the Torelli subgroup in $Out(F_n)$

<u>Fact</u> (Baumslag-Taylor) The kernel

$$T_n := \ker \left[Out(F_n) \to GL_n(\mathbb{Z}) \right]$$

is torsion free. IA_n

<u>Thm</u> (Bestvina-B-Margalit)

1) $cd\, T_n = 2n-4$

2) $H^{2n-4}(T_n)$ is not finitely generated.

Obs:

$$\left\{ \begin{array}{l} x_1 \mapsto [x_n, x_{n-1}]^{k_1}\, x_1\, [x_n, x_{n-1}]^{m_1} \\ x_2 \mapsto [x_n, x_{n-1}]^{k_2}\, x_2\, [x_n, x_{n-1}]^{m_2} \\ \quad \vdots \\ x_{n-2} \mapsto [x_n, x_{n-1}]^{k_{n-2}}\, x_{n-2}\, [x_n, x_{n-1}]^{m_{n-2}} \\ x_{n-1} \mapsto x_{n-1} \qquad x_n \mapsto x_n \end{array} \right\} \cong \mathbb{Z}^{2n-4} \leq T_n$$

Cor: $\mathrm{cd}\, T_n \geq \mathrm{cd}\, \mathbb{Z}^{2n-4} = 2n-4$

Obs: T_n acts freely on the spine K_n of outer space. T_n

$\ulcorner$ $T_n \leq Out(F_n)$ is torsion free.

$Out(F_n)$ acts with finite stabilizers. $\lrcorner$

Thus $Y_n := K_n / T_n$ is a classifying space for T_n.

Cor: $cd\, T_n \leq 2n-3 = \dim K_n = \dim Y_n$

Goal: Y_n is homotopy equivalent to a CW complex of dim $2n-4$.

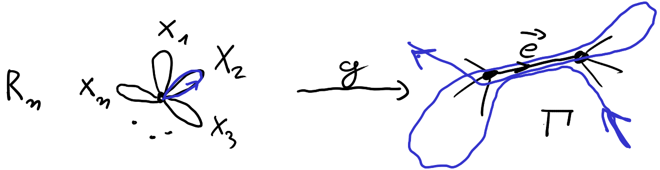
I Y_n as a moduli space

Y_n consists of marked graphs (Γ, g) mod T_n.

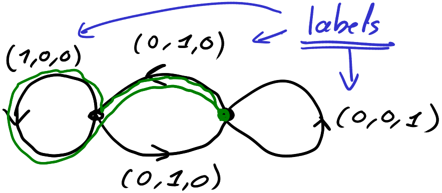
Obs: Two markings belong to the same T_n-orbit if and only if the induce the same homology marking $h: H_1(R_n) \to H_1(\Gamma)$

Thus: Y_n is the geometric realization of the category $H\mathcal{C}_n$ of homologically marked graphs with forest collapses as morphisms.

Visualization (of homology markings)

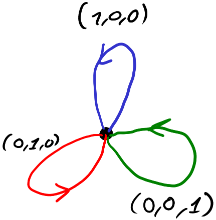


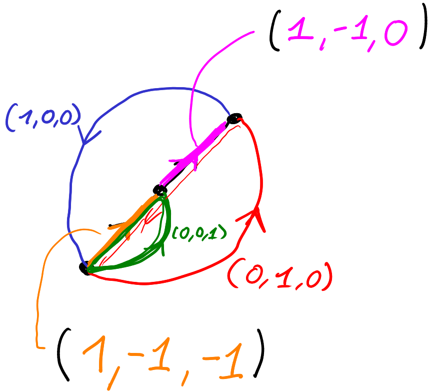
Each petal x_i is mapped to a unique (!) cycle c_i in Γ (no 2-cells!). For an oriented edge $\vec{e}$ let $\vec{e}_i$ be the coefficient of $\vec{e}$ in c_i $\rightsquigarrow$ $(\vec{e}_1, \dots, \vec{e}_n) \in \mathbb{Z}^n$



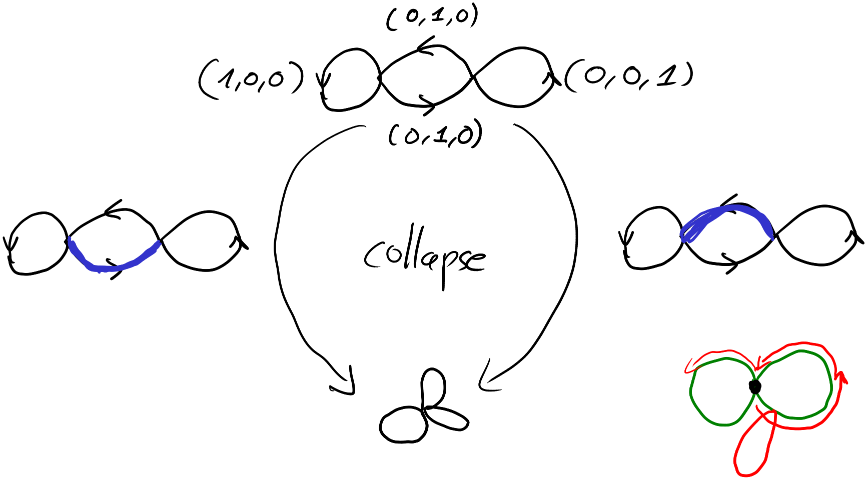
Rem: At each vertex, the sum of incoming labels equals the sum of outgoing labels.
⌜Cycles have 0 boundary⌟

Thus, we can compute the labels for a blow-up: the labels of the original edges stay.





Warning: Y_n is *not* a simplicial complex.



Rem: We think of points in the geometric realization of K_n and Y_n as metric graphs with edge lengths ≤ 1.

A subforest leads to a cube. The orderings of its edges correspond to a simplicial structure on that cube