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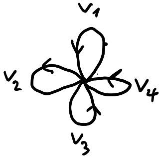
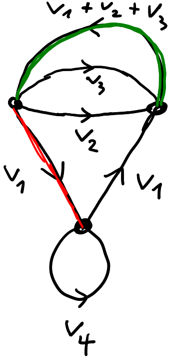
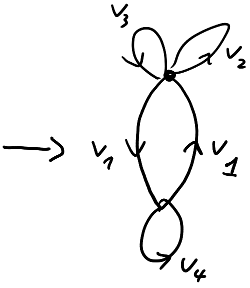
II Morse theory & dimension



Let $\mathcal{S}$ be a rose whose petals are labelled $v_1, \dots, v_n \in \mathbb{Z}^n$. Let Γ be a blow-up. The edges with labels <u>not</u> $\pm v_i$ form a forest. Blowing down this forest shows:





cactus graph

Prop: $st(\mathcal{S})$ is h.e. to the $n-2$ dim complex of cactus graphs

Let ω be an ordinal number and let $(\rho_\alpha)_{\alpha<\omega}$ be a well-ordering of the roses in Y_n.

<u>Def</u>: $Y_{<\beta} := \bigcup_{\alpha<\beta} st(\rho_\alpha)$ ⌜so $Y_n = Y_{<\omega}$⌟

$lk^\downarrow_\beta := Y_{<\beta} \cap st(\rho_\beta)$ ⌜codim $= 1$⌟

<u>Warning</u>: $lk^\downarrow_\beta$ is <u>not</u> the space of directions at ρ_β.

<u>But</u>: $Y_{\leq\beta} := Y_{<\beta} \cup st(\rho_\beta) = Y_{<\beta} \cup_{lk^\downarrow_\beta} st(\rho_\beta)$

Postponed: There is a well-ordering $(S_\alpha)_{\alpha<\omega}$ s.th. each $\mathrm{lk}^{\downarrow}_\beta$ is h.e. to a complex of dim $2n-5$.

Thm: $Y_{<\omega}$ is h.e. to a $(2n-4)$-dim. complex.

Pf. (by transfinite induction)

We construct a family Z_β of CW-complexes of dim. $2n-4$ and homotopy equivalences $h_\beta : Z_\beta \to Y_{<\beta}$ such that $\alpha < \beta \Rightarrow Z_\alpha \subseteq Z_\beta$ and $h_\alpha = h_\beta|_{Z_\alpha}$.

Then $h_\omega : Z_\omega \to Y_{<\omega}$ is what we wanted.

Assume Z_α and $h_\alpha : Z_\alpha \to Y_{<\alpha}$ are given for all $\alpha < \beta$.
To construct Z_β and $h_\beta : Z_\beta \to Y_{<\beta}$ we distinguish:

β is a limit ordinal:

$Y_{<\beta}$ is the ascending union of the $Y_{<\alpha}$ for $\alpha<\beta$

Put $Z_\beta := \bigcup_{\alpha<\beta} Z_\alpha$

h_β is defined by $h_\alpha = h_\beta|_{Z_\alpha}$ for all α.

h_β is a homotopy equivalence (colim & Whitehead)

β is the successor of α:

$$Y_{<\beta} = Y_{\leq\alpha} = Y_{<\alpha} \cup_{\mathrm{lk}^{\downarrow}_{\alpha}} \mathrm{st}(s_\alpha)$$

$$= \operatorname{colim}\left(Y_{<\alpha} \hookleftarrow \mathrm{lk}^{\downarrow}_{\alpha} \hookrightarrow \mathrm{st}(s_\alpha)\right)$$

Choose complexes S_α and L_α of dim $n-2$ and $2n-5$ and maps so that the diagram

$$\begin{tikzcd} Y_{<\alpha} \arrow[d, leftrightarrow, "\text{given}"'] & \mathrm{lk}^{\downarrow}_{\alpha} \arrow[l, hook'] \arrow[r, hook] \arrow[d, leftrightarrow] & \mathrm{st}(s_\alpha) \arrow[d, leftrightarrow] \\ Z_\alpha & L_\alpha \arrow[l] \arrow[r] & S_\alpha \end{tikzcd}$$

becomes homotopy commutative

Define Z_β as the double mapping cone of bot row.

h-colim

Then $\dim Z_\beta \leq 2n-4$.

$$\begin{tikzcd}
Y_{<\alpha} & \mathrm{lk}^{\downarrow}_{\alpha} \arrow[l, hook'] \arrow[r, hook] & st(e_\alpha) \\
Z_\alpha \arrow[u, "h_\alpha"] & L_\alpha \arrow[l] \arrow[r] \arrow[u, "\lambda_\alpha"] & S_\alpha \arrow[u, "\sigma_\alpha"]
\end{tikzcd}$$

The h.e. $h_\beta : Z_\beta \to Y_{\leq \alpha}$ can be chosen to agree with h_α on Z_α, with λ_α on L_α, and with σ_α on S_α.

$\Leftarrow$ the diagram is homotopy commutative. $\blacksquare$

III The Morse function

<u>Def</u>: For $v = (a_1, \dots, a_n) \in \mathbb{Z}^n$ define the <u>norm</u>

$$|v| := (|a_1|, |a_2|, \dots, |a_n|) \in \mathbb{Z}_{\geq 0}^n \quad \text{(lex. order)}$$

For a matrix:

$$M = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \rightsquigarrow |M| := (|v_n|, \dots, |v_1|) \in \left(\mathbb{Z}_{\geq 0}^n\right)^n \quad \text{(lex. order)}$$

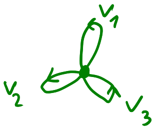
<u>Obs</u>: $|u| = |v| \Rightarrow u = v \mod 2$

$|v_i| = |v_j|,\ i \neq j \Rightarrow M$ not invertible $\mod 2$

Def: For a labelled rose S, the norm $|S|$ is defined as $(|v_n|, \ldots, |v_1|)$ where v_i are the labels of the petals of S ordered such that $|v_n| < \cdots < |v_1|$.

I.e.: $|S| = \left| \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \right|$ for the ordering $|v_n| < \cdots |v_1|$.

Exercise: Adjacent roses have different norms.



Def: $$\mathrm{lk}(\varrho) := \mathrm{st}(\varrho) \cap \bigcup_{\varrho' \neq \varrho} \mathrm{st}(\varrho')$$

$$\mathrm{lk}^{\downarrow}(\varrho) := \mathrm{st}(\varrho) \cap \bigcup_{|\varrho'| < |\varrho|} \mathrm{st}(\varrho') \quad \subseteq \mathrm{lk}(\varrho)$$

Obs: Let ϱ be a rose with labels $|v_n| < \cdots < |v_1|$. A blow-up Γ lies in $\mathrm{lk}(\varrho)$ if it has a fully blown up edge not labelled $\pm v_i$. If one of those edges has a label u with $|u| < |v_1|$, then $\Gamma \in \mathrm{lk}^{\downarrow}(\varrho)$.

$\mathrm{st}(\varsigma) = \{$ blow ups of $\varsigma\}$

$\Gamma \in \mathrm{st}(\varsigma) \iff \varsigma$ can be obtained from Γ by blowing down a forest.

Def: An edge is descending, if its label has smaller norm than at least one petal label, i.e.: smaller than $|v_1|$.

Def: A blow-up T is completely descending if all blown up edges are descending.

Note: $lk^{\downarrow}(\sigma)$ def. retracts onto $lk^{\Downarrow}(\sigma)$ by collapsing the forest of non-descending blown up edges.

<u>Rem</u>: Let $\Gamma \in lk^{\Downarrow}(s)$ be fully blown up, i.e. all edges have length 1. We can associate to Γ a polysimplicial cell C_Γ

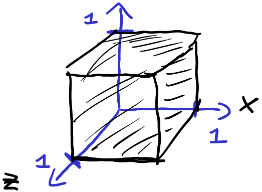
$$C_\Gamma = \Delta_1 \times \cdots \times \Delta_n \times \Delta_*$$

$\Delta_i \leftrightsquigarrow$ petal label v_i

$\Delta_* \leftrightsquigarrow$ all other labels

$\rightsquigarrow$ C_Γ has dim (# vertices of Γ) - 2

$\rightsquigarrow$ CW structure on $lk^{\Downarrow}(s)$



$$\Delta^{(n)} = \{(x_0, \dots, x_n) \in [0,1]^{n+1} \mid \exists i : x_i = 1\}$$

Δ_i : Δ (edges with label $\pm v_i$)

Δ_* : Δ (edges not labelled $\pm v_1$ or $\pm v_2, \dots$)

$$\dim C_\Gamma = \dim(\Delta_1 \times \cdots \times \Delta_n \times \Delta_*) = \# \mathrm{vert}(\Gamma) - 2$$

C_Γ has top. dim $\Longleftrightarrow$ Γ is trivalent

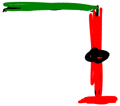
$$\dim(C_\Gamma) = 2n - 4$$

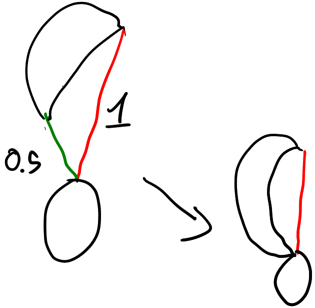
$\boxed{\text{in } \mathbb{k}\,!}$

$|v_2+v_3| < |v_1|$

Γ

$\rightsquigarrow \quad C_\Gamma = \Delta_2 =$





Key Lemma: Let S be a rose with labels $v_1, \dots, v_n$ and $|v_n| < \dots < |v_1|$. If $v_1 = (|a_1^1|, \dots, |a_1^n|)$

$$v_1 + \sum_{i=2}^{n} a_i v_i \qquad a_i \in \{-1, 0, 1\}$$

is desending, then $|v_1 + a_2 v_2 + \dots + a_n v_n| < |v_1|$

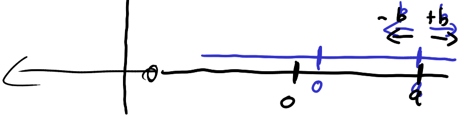
$$v_1 - \sum a_i v_i \qquad \text{(opposite label)}$$

is not.

$|a+b| \leq |a| \Longrightarrow b = 0$

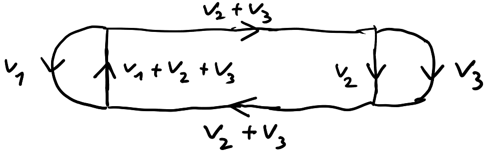
or

$|a - b| > |a|$



pf on layer 2

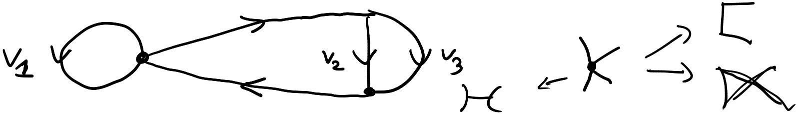
Example:

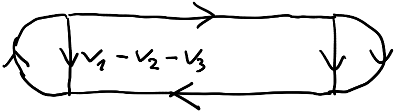


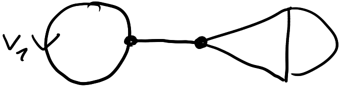
$=: \Gamma \in lk^{\Downarrow}(\sigma)$

$|v_1|$ max norm

Claim: C_Γ has a free face, namely:







this is not completely descending (key lemma)

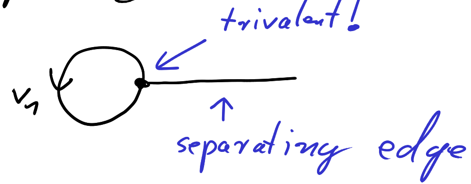
separating edge $\rightsquigarrow$ no adjacent cell

pf on layer 2

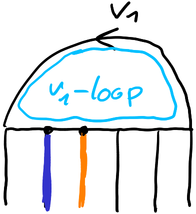
Thm: Let L be obtained from $lk^{\downarrow}(s)$ by pushing in free faces until no longer possible. Then L does not contain a top-dim. cell, i.e. a C_Γ for a trivalent blow-up Γ.

pf: Consider a C_Γ so that the v_1-loop in Γ has minimal length ℓ

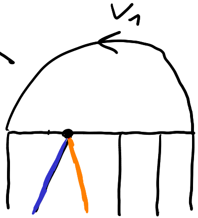
1) $\ell > 1$ otherwise

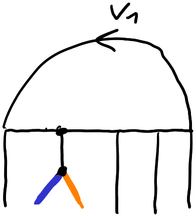


2)



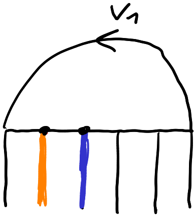
in L by assumption





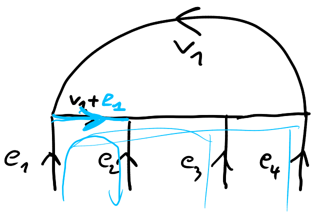
not in L: v_1-loop too short. ↯

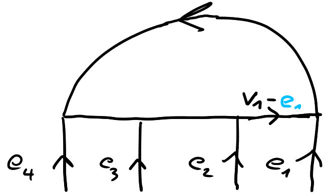
swap order of two "adjacent" edges



in L because otherwise free face

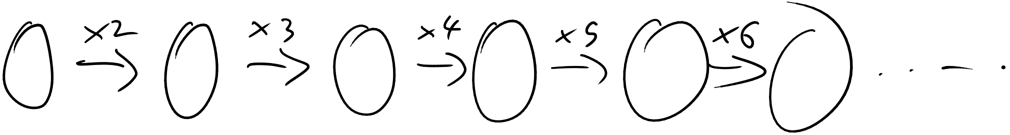
3) Use the swaps to build a gallery from C_Γ to C_Δ where the cyclic order of the edges issuing from the v_1-loop is reversed:





not in $lk^{\Downarrow}(\sigma)$ by key lemma

$\boxtimes$



mapping telescope $\not\simeq$ 1-dim.

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Rem: Put $\Omega := \{ \mathfrak{s} : \text{labeled rose} \}$.

Extend $\mathfrak{s} \subset \mathfrak{s}' :\Leftrightarrow |\mathfrak{s}| < |\mathfrak{s}'|$ to a total order $<$ on Ω.

Obs: $<$ is a well-ordering on Ω.

⌜ $\mathbb{Z}_{\geq 0}^n$ is well-ordered.

The set $(\mathbb{Z}_{\geq 0}^n)^n$ of norms is well-ordered.

Each norm is the norm of fin. many roses. ⌟

Let ω be the order type of $(\Omega, <)$.