Let $S$ be a rose whose petals are labelled $v_1, \ldots, v_n \in \mathbb{Z}^m$. Let $\Gamma$ be a blow-up. The edges with labels $\not\equiv \pm v_i$ form a forest. Blowing down this forest shows $$\text{Prop: } \text{st}(S) \text{ is b.e. to the n-2 dim complex of cactus graphs}$$
Let $\omega$ be an ordinal number and let $(S_\alpha)_{\alpha<\omega}$ be a well-ordering of the roses in $Y_\alpha$.

**Def:** $Y_\beta := \bigcup_{\alpha<\beta} st(S_\alpha)$  
\[ \text{so } Y_\beta = X_{<\omega} \]

**Def:** $1k_\beta^\uparrow := Y_\beta \setminus st(S_\beta)$  
\[ \text{codim} = 1 \]

**Warning:** $1k_\beta^\uparrow$ is not the space of directions at $s_\alpha$.

**But:** $Y_{\leq \beta} := Y_{<\beta} \cup st(S_\beta) = Y_{<\beta} \cup 1k_\beta^\uparrow st(S_\beta)$
Postponed: There is a well-ordering $(S_\alpha)_{\alpha<\omega}$ s.t. each $\text{lk}_\beta$ is h.e. to a complex of dim $2n-5$.

**Thm:** $Y_{<\omega}$ is h.e. to a $(2n-4)$-dim. complex.

**pf:** (by transfinite induction)

We construct a family $Z_\beta$ of CW-complexes of dim. $2n-4$ and homotopy equivalences $h_\beta : Z_\beta \to Y_{<\beta}$ such that $\alpha < \beta \implies Z_\alpha \leq Z_\beta$ and $h_\alpha = h_\beta|Z_\alpha$.

Then $h_\omega : Z_\omega \to Y_{<\omega}$ is what we wanted.
Assume $Z_\alpha$ and $h_\alpha : Z_\alpha \to Y_{<\alpha}$ are given for all $\alpha < \beta$.

To construct $Z_\beta$ and $h_\beta : Z_\beta \to Y_{<\beta}$ we distinguish:

- $\beta$ is a limit ordinal:

  $Y_{<\beta}$ is the ascending union of the $Y_{<\alpha}$ for $\alpha < \beta$

  Put $Z_\beta := \bigcup_{\alpha < \beta} Z_\alpha$

- $h_\beta$ is defined by $h_\alpha = h_\beta |_{Z_\alpha}$ for all $\alpha$

- $h_\beta$ is a homotopy equivalence (colim & Whitehead)
\( \beta \) is the successor of \( \alpha \):

\[
\chi_\beta = \chi_\alpha = \chi_\alpha \cup \mathbf{lk}_\alpha \xrightarrow{\text{st}} (S_\alpha)
\]

\[
= \text{colim} \left( \chi_\alpha \xrightarrow{\text{lk}_\alpha} \xrightarrow{\text{st}} (S_\alpha) \right)
\]

Choose complexes \( S_\alpha \) and \( L_\alpha \) of \( \text{dim} \ 2n-2 \) and \( 2n-5 \) and maps so that the diagram becomes homotopy commutative.

Define \( Z_\beta \) as the double mapping cone of bot rows.

Then \( \text{dim} \ Z_\beta \leq 2n-4 \).

\( h\text{-colim} \)
The h.e. \( h_\beta : Z_\beta \to Y_{\leq \alpha} \) can be chosen to agree with \( h_\lambda \) on \( Z_\lambda \), with \( \lambda_\lambda \) on \( L_\lambda \), and with \( \sigma_\alpha \) on \( S_\alpha \).

\[
\begin{align*}
Y_\alpha & \leftarrow L_{k_{\alpha}} \rightarrow \text{st}(\xi_{\alpha}) \\
\uparrow h_\alpha & \uparrow \lambda_\alpha & \uparrow \sigma_\alpha \\
Z_{\alpha} & \leftarrow L_{\alpha} \rightarrow S_{\alpha}
\end{align*}
\]

\( \iff \) the diagram is homotopy commutative. \( \square \)
III The Morse function

Def: For \( v = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) define the norm

\[ |v| := (|a_1|, |a_2|, \ldots, |a_n|) \in \mathbb{Z}_{\geq 0}^n \] (lex. order)

For a matrix:

\[ M = \begin{pmatrix} \vdots \\ v_m \end{pmatrix} \implies |M| := (|v_{m1}|, \ldots, |v_{m1}|) \in (\mathbb{Z}_{\geq 0}^n)^n \] (lex. order)

Obs: \( |w| = |v| \implies w = v \mod 2 \)

\[ |v_{i1}| = |v_{j1}|, i \neq j \implies M \text{ not invertible } \mod 2 \]
Def: For a labelled rose $s$, the norm $|s|$ is defined as $(|v_m|, \ldots, |v_1|)$ where $v_i$ are the labels of the petals of $s$ ordered such that $|v_m| < \cdots < |v_1|$.

I.e.: $|s| = |\binom{m}{2}|$ for the ordering $|v_m| < \cdots < |v_1|$.

Exercise: Adjacent roses have different norms.
Def: \[ lk(s) := s^+(s) \cup \bigcup_{s' \neq s} s^+(s') \]
\[ lk'(s) := s^+(s) \cup \bigcup_{1|s'| < 1|s|} s^+(s') = lk(s) \]

Obs: Let \( s \) be a rose with labels \( |v_n| < \cdots < |v_1| \).

A blow-up \( T \) lies in \( lk(s) \) if it has a fully blown up edge not labelled \( \pm v_i \). If one of those edges has a label \( u \) with \( |u| < |v_i| \), then \( T \in lk'(s) \).
\[ st(s) = \{ \text{blow ups of } \mathcal{E} \} \]

\[ \forall \mathcal{E} \in st(s) \iff s \text{ can be obtained from } \mathcal{E} \text{ by blowing down a forest}. \]
Def: An edge is descending, if its label has smaller norm than at least one petal label, i.e. smaller than $1_{\mathbf{v}_1}$.

Def: A blow-up $T$ is completely descending if all blown up edges are descending.

Note: $1k^v(s)$ def. retracts onto $1k^v(s)$ by collapsing the forest of non-descending blown up edges.
Rem: Let $\Pi \in \Omega^k(s)$ be fully blown up, i.e. all edges have length 1. We can associate to $\Pi$ a polysimplicial cell $C_{\Pi}$

$$C_{\Pi} = \Delta_1 \times \cdots \times \Delta_m \times \Delta_x$$

$\Delta_i \leftrightarrow$ petal label $v_i$

$\Delta_x \leftrightarrow$ all other labels

$\Delta^{(m)} = \{(x_0, \ldots, x_m) \in [0,1]^{m+1} \mid \exists i : x_i = 1\}$

$\implies C_{\Pi}$ has dim $(\# \text{vertices of } \Pi) - 2$

$\implies$ CW structure on $\Omega^k(s)$
\( \Delta_i : \Delta \) (edges with label \( \pm v_i \))

\( \Delta_x : \Delta \) (edges not labelled \( \pm v_1, \ldots \))

\[ \dim C_{\Gamma} = \dim(\Delta_1 \times \cdots \times \Delta_n \times \Delta_x) = \# \text{vert}(\Gamma) - 2 \]

\( C_{\Gamma} \) has top. dim \( \iff \) \( \Gamma \) is trivalent

\[ \dim(C_{\Gamma}) = 2n - 4 \]
$|v_2 + v_3| < |v_1|$

$\nabla \Rightarrow C_\Pi = \Lambda_2 = \nabla$

0.5

$1$
Key Lemma: Let $S$ be a rose with labels $V_1, \ldots, V_n$ and $|V_n| < \cdots < |V_1|$. If \( V_1 = (a_1, \ldots, a_n) \)

\[ V_1 + \sum_{i=2}^{n} a_i V_i \]

is descending, then

\[ |V_1 + a_2 V_2 + \cdots + a_n V_n| < |V_1| \]

is not.

\[ |a + b| \leq |a| |b| \Rightarrow |a| = |b| = 0 \]

or

\[ \frac{a}{b} + \frac{b}{a} \]

or

\[ |a + b| = |b| \]

or

\[ \|a\| > \|b\| \]

pf on layer 2.
Example: \( \Gamma \in \mathbb{K}^k(\mathcal{S}) \) with \( v_1 \) max norm

Claim: \( \Gamma \) has a free face, namely:

\[
\begin{align*}
v_2 & \\ \leadsto & \\Rightarrow \\
\end{align*}
\]

this is not completely descending (key lemma)

\[
\begin{align*}
v_1 - v_2 - v_3 & \\
\end{align*}
\]

separating edge \( \Rightarrow \) no adjacent cell
Thm: Let $L$ be obtained from $\mathcal{K^g}(s)$ by pushing in free faces until no longer possible. Then $L$ does not contain a top-dimensional cell, i.e. a $C_7$ for a trivalent blow-up $\Pi$.

\[ \text{pf: Consider a } C_7 \text{ so that the } v_1 \text{-loop in } \Pi \text{ has minimal length } \ell \]

1. $\ell > 1$ otherwise $v_1$ is a trivalent vertex.
2) Swap order of two "adjacent" edges.

- In $L$ by assumption.

Not in $L$: $v_1$-loop too short.

- In $L$ because otherwise free face.
3) Use the swaps to build a gallery from $C_P$ to $C_{\Delta}$ where the cyclic order of the edges issuing from the $v_1$-loop is reversed.

\[ \text{not in } \text{lk}^*(S) \text{ by key lemma} \]
mapping telescope \neq 1\text{-dim.}
Rem: Put \( \Omega := \{ s : \text{labeled rose} \} \).

Extend \( s \preceq s' \iff |s| < |s'| \) to a total order \( \prec \) on \( \Omega \).

Obs: \( \prec \) is a well-ordering on \( \Omega \).

\( \mathbb{Z}_{\geq 0} ^{\omega} \) is well-ordered.

The set \( (\mathbb{Z}_{\geq 0} ^{\omega})^{\omega} \) of norms is well-ordered.

Each norm is the norm of fin. many roses.

Let \( \omega \) be the order type of \( (\Omega, \prec) \).