BOREL-SERRE DUALITY

Mikala Ørsnes Jansen
University of Copenhagen

MASTERCLASS

High dimensional cohomology of moduli spaces.

SETTING THE SCENE

We are interested in

$$\Gamma \leqslant SL_n(\mathbb{Z})$$

finite index
torsion free

eg. congruence subgroups

Γ acts freely and properly discontinuously on

$$X = SO(n) \backslash SL_n(\mathbb{R})$$

X/Γ is
* a smooth manifold
* a model for $B\Gamma$.

but $\boxed{\text{not compact!}}$

THE BOREL-SERRE COMPACTIFICATION

Introduced in 1973

$$X/\Gamma \xhookrightarrow{\sim} \overline{X}/\Gamma$$

* compact
* a smooth manifold with corners
* a model for $B\Gamma$.

get back to this

Idea:

* Partial compactification $X \subseteq \overline{X}$
* Extend action of Γ to $\overline{X}$.

Applications:

* Algebraic K-theory (Borel, Quillen)
* Borel-Serre duality

DUALITY GROUPS (Bieri–Eckmann)

dualising module

Definition

G is a duality group of dim. N wrt. a right G-module $\mathcal{D}$ if there is $e \in H_N(G;\mathcal{D})$ such that

$$(e \cap -) : H^k(G;A) \xrightarrow{\cong} H_{N-k}(G;\mathcal{D} \otimes_{\mathbb{Z}} A)$$

for all left G-modules A and all $k \in \mathbb{Z}$.

"Poincaré duality with a twisting of the coefficients".

STEINBERG MODULE

Let S_n denote the poset of proper $\neq 0$ subspaces in $\mathbb{Q}^n$.

The Tits building is the simplicial complex

$$\boxed{T_n = |N_\bullet(S_n)|}$$

p-cell in T_n corresponds to a flag
$0 \subsetneq V_0 \subsetneq \cdots \subsetneq V_p \subsetneq \mathbb{Q}^n$.

Solomon-Tits: $T_n \simeq \bigvee S^{n-2}$

Definition

The Steinberg module is the $SL_n(\mathbb{Z})$-module

$$St_n := \tilde{H}_{n-2}(T_n; \mathbb{Z})$$

BOREL-SERRE DUALITY

Theorem (Borel-Serre 73)

Γ is a duality group of dim $\binom{n}{2}$ wrt. the Steinberg module St_n.

$$H^k(\Gamma; A) \xrightarrow{\cong} H_{\binom{n}{2}-k}(\Gamma; St_n \otimes A)$$

for all A and all k.

Corollary:

* $\operatorname{cd} \Gamma = \operatorname{vcd} SL_n\mathbb{Z} = \binom{n}{2}$
* $H^{\binom{n}{2}}(\Gamma; A) \cong (St_n \otimes A)_\Gamma$

PROOF OF BOREL-SERRE DUALITY

finite G-proj. res of $\mathbb{Z}$

Theorem (Bieri-Eckmann '73)

G group of type FP.

If there is an $N \geq 0$ such that

$$H^k(G;\mathbb{Z}G) = \begin{cases} \text{free abelian} & k = N \\ 0 & \text{else} \end{cases}$$

then G is a duality group of dim N

wrt. $\mathcal{D} = H^N(G;\mathbb{Z}G)$.

Γ is of type FP (FL)

$\overline{X}/\Gamma \simeq \underbrace{K(\Gamma,1)}$ finite cw-cpx

Existence of

PROOF OF BOREL-SERRE DUALITY

We have $d = \dim \overline{X} = \boxed{\binom{n}{2}+n-1}$ and

$$H^i(\Gamma; \mathbb{Z}\Gamma) \cong H^i_c(\overline{X})$$

$K(\Gamma,1)$ finite CW-cpx

PLD $\cong H_{d-i}(\overline{X}, \partial\overline{X})$

LES of pair $\cong \tilde{H}_{d-i-1}(\partial\overline{X}) \cong \tilde{H}_{d-i-1}(T_n)$

$* \simeq X \overset{\sim}{\hookrightarrow} \overline{X}$

Main technical ingredient: $\partial\overline{X} \simeq T_n$

By the Solomon-Tits theorem,

$$H^i(\Gamma; \mathbb{Z}\Gamma) = \begin{cases} \text{free abelian} & i = \binom{n}{2} \\ 0 & \text{else} \end{cases}$$

PROOF OF BOREL-SERRE DUALITY

Theorem (Bieri–Eckmann 73)

Γ group of type FP. ✓ $\binom{n}{2}$

If there is an $N \geq 0$ such that

$$H^k(\Gamma;\mathbb{Z}\Gamma) = \begin{cases} \text{free abelian} & k=N \;✓ \\ 0 & \text{else} \;✓ \end{cases}$$

then Γ is a duality group of dim N wrt. $\mathcal{D} = H^N(\Gamma;\mathbb{Z}\Gamma)$.

$$\begin{array}{c} \wr\!\| \\ \tilde{H}_{n-2}(T_n) =: St_n \end{array}$$

Assuming $\partial \overline{X} \simeq T_n$ $(\square)$

CONSTRUCTION OF $\overline{X}$

$P(\mathbb{R})$

Parabolic subgroups in SL_n

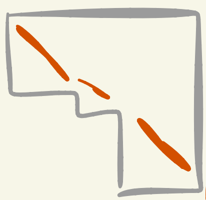
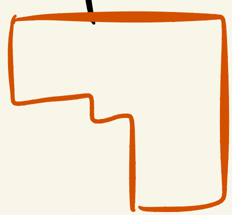
Block upper triangular

$P =$ 

$\leq$  $= SL_n$ $\left(+ SL_n(\mathbb{Z}) \text{ conj.} \right)$

Geodesic action

For parabolic P, consider $A_P \leq P$ of diagonal matrices such that each diagonal block is positive scalar.

$A_P =$  $\leq$  $= P$

Note: $A_P \cong \mathbb{R}_{>0}^{k_P} \leftarrow \text{rank } P = \#\text{blocks} - 1$

Define: $A_P \curvearrowright X$ left multiplication on

$$X = SO(n)\backslash SL_n(\mathbb{R}) \cong (P \cap SO(n))\backslash P$$

Borel-Serre corners

Set $\boxed{e_P := A_P \backslash X \cong \mathbb{R}^{d-k_P}}$

Then $X \cong A_P \times e_P$

partially compactify the orbits!

$$\boxed{A_P \cong \mathbb{R}^{k_P}_{>0} \longrightarrow \mathbb{R}^{k_P}_{\geq 0} =: \overline{A_P}}$$

Define

$$\boxed{\begin{aligned} X(P) &:= \overline{A_P} \times_{A_P} X \\ &\cong \overline{A_P} \times e_P \\ &\cong \mathbb{R}^{k_P}_{\geq 0} \times \mathbb{R}^{d-k_P} \end{aligned}}$$

($\cong$  $\times$ euclidean)

CONSTRUCTION OF $\overline{X}$

For all $P \leq Q$, we get compatible embeddings

$$\boxed{X(Q) \hookrightarrow X(P)}$$

which respect the corner structure

$$X(P) \cong \mathbb{R}^{k_P}_{\geq 0} \times \mathbb{R}^{d-k_P}$$

Glue corners

$$\boxed{\overline{X} = \bigcup_P X(P)}$$

manifold with corners

EXAMPLE: $SL_3(\mathbb{Z})$ $X \cong \mathbb{R}^5$

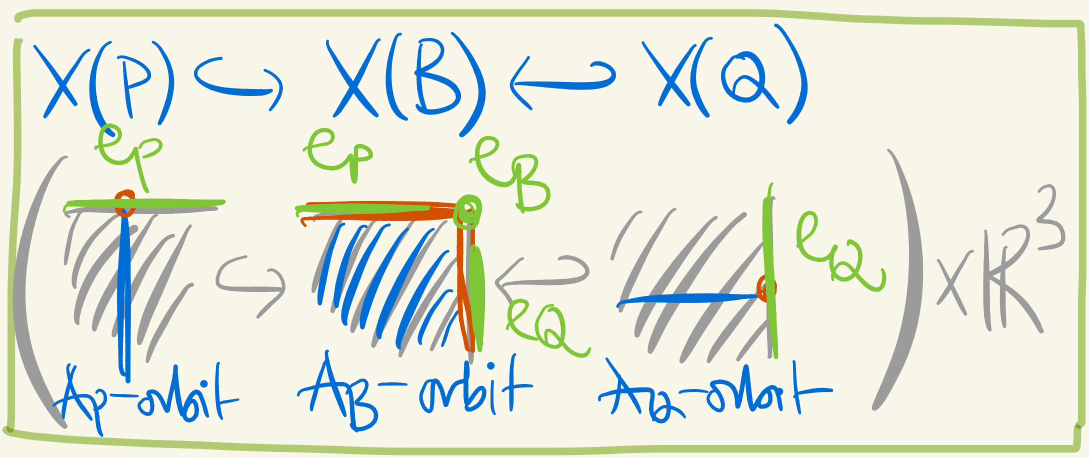
Parabolic subgroups: $P \geq B \leq Q$

Corners:

$$X(B) \cong \overline{A}_B \times e_B \cong \mathbb{R}^2_{\geq 0} \times \mathbb{R}^3_{>0}$$

$$X(P) \cong \overline{A}_P \times e_P \cong \mathbb{R}_{\geq 0} \times \mathbb{R}^4_{>0}$$

$$X(Q) \cong \overline{A}_Q \times e_Q \cong \mathbb{R}_{\geq 0} \times \mathbb{R}^4_{>0}$$



BOUNDARY COMPONENTS

Partition of $\overline{X} = \bigcup X(P)$.

boundary component

Identify $e_P = A_P \backslash X \subseteq \overline{X}$ by

$$e_P \underset{\text{at origin}}{\hookrightarrow} \overline{A_P} \times e_P \cong X(P) \subseteq \overline{X}.$$

$\overline{A_P} \,\|\, \mathbb{R}_{\geq 0}^{k_P}$

Then $\boxed{\partial\overline{X} = \coprod_{P \text{ proper}} e_P \qquad \text{~~with } e_G = X~~}$

$G = SL_n$

and $\boxed{\overline{e_Q} = \coprod_{P \leq Q} e_P}$

$\partial\overline{X}$ "encodes" the poset of proper parabolic subgroups.

PARABOLIC SUBGROUPS $\longleftrightarrow$ FLAGS

$\mathcal{Q}$ set of maximal parabolic subgroups.

Theorem (Borel–Serre 73)

$\mathcal{F} = \{\overline{e_Q}\}_{Q\in\mathcal{Q}}$ is a locally finite cover of $\partial\overline{X}$ whose nerve is

* homotopy equiv. to $\partial\overline{X}$.
* the Tits building.

p-cells in $|N_\bullet(\mathcal{F})|$ and $T_n = |N_\bullet(S_n)|$

$Q_0, \dots, Q_p \in \mathcal{Q}$ s.t. $\bigcap_i \overline{e_{Q_i}} \neq \emptyset$

$\underset{\|}{}$ e_P

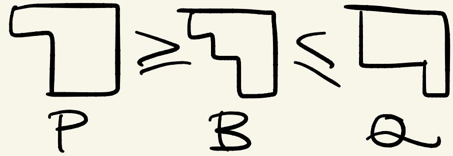
$\bigcap_i Q_i = P$ parabolic

$0 \subsetneq V_0 \subsetneq \dots \subsetneq V_p \subsetneq \mathbb{Q}^n$

flag

in bijection (stabilisers).

EXAMPLE : $SL_3(\mathbb{Z})$



$\partial\overline{X}$ glued together out of

e_P $\quad$ e_B

e_Q

via $SL_3(\mathbb{Z})$-conjugation

T_n glued together out of

$\langle e_1 \rangle$

$\langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle$

$\langle e_1, e_2 \rangle$

via $SL_3(\mathbb{Z})$-action on $\mathbb{Q}^3$

BOREL-SERRE DUALITY

Theorem (Borel-Serre 73)

Any finite index torsion-free $\Gamma \leq SL_n(\mathbb{Z})$ is a duality group of dim $\binom{n}{2}$ wrt. the Steinberg module St_n.

In other words, there is a class $e \in H_{\binom{n}{2}}(\Gamma; St_n)$ such that

$$H^k(\Gamma; A) \xrightarrow[e\cap -]{\cong} H_{\binom{n}{2}-k}(\Gamma; St_n \otimes A)$$

for any Γ-module A and all $k \in \mathbb{Z}$.

FINAL REMARKS

This is all done in greater generality:

* Virtual duality groups.

$SL_n(\mathbb{Z})$ (torsion in group
vs.
$\mathbb{Q}$ torsion in coefficient modules)

* S-arithmetic groups, not just $SL_n(\mathbb{Z})$.

$GL_n(\mathbb{Z})$ (orientation module)

* Over number fields, not just $\mathbb{Q}$.

THE END