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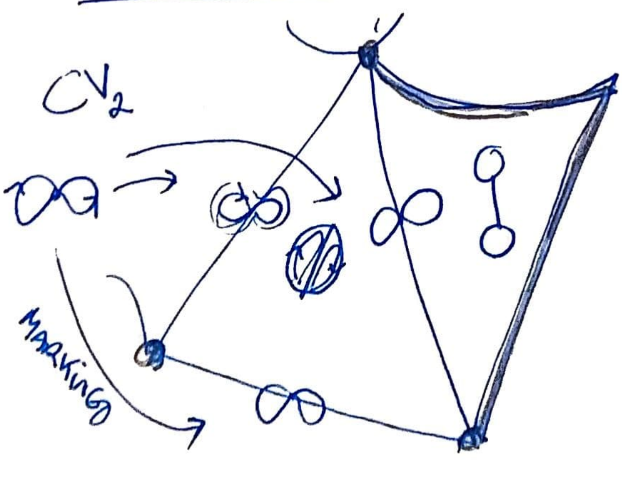
Free factor cpx as boundary structures

jt with Radhika Gupta ('20)

Group	Moduli space	∂ subspaces of $\mathbb{Z}^n$ " = " Tits building
$SL_n\mathbb{Z}$, $Sp_{2n}\mathbb{Z}$ ↻ $\mathbb{Z}^n$	Sym. space $\simeq *$	Tits building $\simeq \vee S^{n-2}$
$Mod_g = Out(\pi_1 S_g)$ ↻ $(\pi_1)(S_g)$	~~Teichmüller space~~ Teichmüller space $\simeq *$	curve cpx " = syst. of curves on S_g" $\simeq \vee S^{2g-2}$
$Out F_n$ ↻ F_n	Outer space $\simeq *$	$\boxed{? \simeq ?}$ missing piece

1. Simplicial bdry of unreduced outer space CV_n

unreduced i.e. allowing separating edges.



<u>Note:</u> unreduced outer space is homotopic to the reduced one, but their bdry are different.

$CV_n \cup \partial CV_n$ = moduli space of marked metric graphs, where subgraphs may have length 0.

$= \{$ MARKED METRIC GRAPHS OF GROUPS $\}$

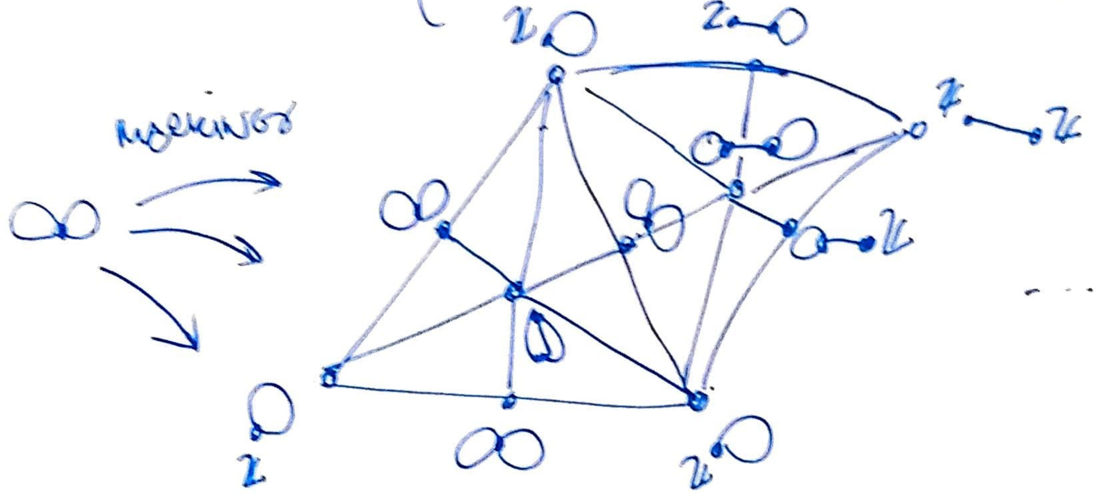
↑ REMEMBERING π_1(COLLAPSED GRAPH) WITH IDENTIFICATION COMING FROM MARKING.

SUBDIVISION

$\cong |\mathcal{F}\mathcal{S}_n|$

POSET OF FREE SPLITTINGS OF F_n

$= \left(\{\text{MARKED GRAPHS OF GROUPS}\}, \text{COLLAPSE SUBGRAPHS} \right)$



$\mathcal{F}\mathcal{S}_n = L_n \cup \mathcal{F}\mathcal{S}_n^\infty$ WHERE

$L_n =$ SPINE OF CV_n

$= \{ S \in \mathcal{F}\mathcal{S}_n \mid$ EVERY GROUP IS TRIVIAL $\}$

$\mathcal{F}\mathcal{S}_n^\infty = \mathcal{F}\mathcal{S}_n \setminus L_n$

(ANALOGUE OF A^∞ FROM TARA'S TALK)

⚠ $|\mathcal{F}\mathcal{S}_n| \neq |L_n| \cup |\mathcal{F}\mathcal{S}_n^\infty|$

FREE FACTOR COMPLEXES AT ∞

Def: $A \subseteq F_n$ FREE FACTOR IF $F_n = A * B$

FREE FACTOR CPX $\mathcal{F}_n$ = POSET OF CONJUGACY CLASSES OF FREE FACTORS IN F_n.

BECAUSE WANT $Out(F_n)$-ACTION.

Def (HANDEL-MOSHER)

- A FREE FACTOR SYSTEM IN F_n IS A COLLECTION $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$ OF CONJUGACY CLASSES OF FREE FACTORS S.T. $F_n = A_1 * \cdots * A_k * B$.
- FOR FREE FACTOR SYSTEMS $\mathcal{A}, \mathcal{A}'$, WE'LL SAY $\mathcal{A} \leq \mathcal{A}'$ IF $\forall [A_i] \in \mathcal{A}$, $\exists [A_j'] \in \mathcal{A}'$ S.T. $A_i \subseteq A_j'$.

THE CPX OF FREE FACTOR SYSTEMS $\mathcal{FF}_n$ IS THE ASSOCIATED POSET.

E.g. IN $F_3 = \langle a,b,c \rangle$ $\quad \{[\langle a \rangle], [\langle b \rangle]\} \leq \{[\langle a,b \rangle]\}$
$\leq \{[\langle a,b \rangle], [\langle c \rangle]\}$

$\dim(\mathcal{FF}_n) = 2n-3$

$\mathcal{F}_n \subset \mathcal{FF}_n$

WANT: $\mathcal{FS}_n^\infty \xrightarrow{f} \mathcal{FF}_n$

$S \longmapsto$ COLLECTION OF NON-TRIVIAL VERTEX GROUPS.

<u>THM</u> (B – Gupta) 1) $f : \partial\mathcal{S}_n^\infty \longrightarrow \partial\mathcal{J}_n$ is a htpy equivalence.

2) f restricts to a htpy equi $\partial\mathcal{S}_n^1 \longrightarrow \mathcal{J}_n$

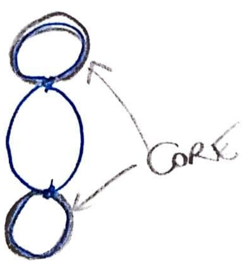
$$\partial\mathcal{S}_n^1 = \{\text{exactly 1 non-trivial vertex group}\}.$$

<u>PF</u>: The fibers are contractible.
↑ Relative versions of $\partial\mathcal{S}_n \simeq *$ [HATCHER]

3. Connectivity of $\mathcal{J}_n$, $\partial\mathcal{J}_n$

<u>Def</u>: A <u>core subgraph</u> of a graph G is a subgraph H s.t. the fundamental gp of every comp. is non-trivial and s.t. every vertex has valence ≥ 2 in the subgraph.

<u>EX</u>:



Let $Z \subset L_n \times \partial\mathcal{S}_n^\infty$ "neighborhood of $\partial\mathcal{S}_n^\infty$"

$= \{(G, S) \mid S$ is obtained by collapsing a core subgraph of $G\}$

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ L_n & & \partial\mathcal{S}_n^\infty \end{tikzcd}$$

<u>LEMMA 1</u>: p_2 is a htpy equiv

<u>LEMMA 2</u>: $\forall\, G \in L_n$, the fiber $p_1^{-1}(L_{\geq G}) = (L_n)_{p_1}$ is $(n-3)$-connected.

PF IDEA: $p_1^{-1}(G) = \{ S \in \mathcal{F}S_n^{\Delta} \mid G \text{ COLLAPSES TO } S \}$

$\cong \{ \text{PROPER CORE SUBGRAPH OF } G, \subseteq \}$

FINITE POSET HIGHLY CONN.

($\simeq *$ EXACTLY IF $\exists$ SEPARATING EDGE)

COR: Z is $(n-3)$-CONN.

$\wr$

$\mathcal{F}S_n^{\infty} \simeq \mathcal{F}\mathcal{F}_n$

(IN FACT $(n-2)$-CONNECTED) BUT dim $2n-3$ ...

DON'T KNOW ANY NON-TRIVIAL CLASS!

COR: $\mathcal{F}_n \simeq \mathcal{F}S_n^1$ is $(n-3)$-CONN.

$\simeq \bigvee S^{n-2}$