High-dimensional cohomology of moduli spaces
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Review: Homological algebra and (co)homology of groups

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This review packages assumes familiarity with the following subjects:

- The basic language of category theory
- The basic theory of modules over a ring
- Projective modules
- The basic theory of (co)chain complexes and their homology, chain maps and chain homotopies, the long exact sequence on homology induced by a short exact sequence of chain complexes.
- Simplicial, cellular, and singular (co)homology of spaces
- The basic theory of fundamental groups and covering spaces
- Tensor products over a (possibly noncommutative) ring, universal property and right-exactness
- Hom functors and left-exactness
- Representations of a group G, and relationship to the group algebra

Asterisks indicate more advanced exercises (or exercises that employ spectral sequences). The reader may wish to skip these at first reading.

Please help me polish these notes by sending corrections to `jchw@umich.edu`!

Contents

1 Review: Projective resolutions

Throughout these notes we assume all rings have unit 1.

1.1 Projective R-modules

We first review the equivalent definitions of a projective module.

Exercise 1. Let R be a ring and P an R-module. Each of the folllowing statements may be taken as the defining statement that P is *projective*. Prove that these statements are equivalent.

(i) For every surjective R-module homomorphism $d : N \to M$ and every R-module homomorphism $f : P \to M$, there is a (not necessarily unique) homomorphism $g : P \to N$ such that $d \circ h = f$.

$$\begin{tikzcd} P \arrow[d, dotted, "\exists g"'] \arrow[dr, "f"] & & \\ N \arrow[r, two heads, "d"'] & M \arrow[r] & 0 \end{tikzcd}$$

(ii) Any short exact sequence of R-modules of the following form is split

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0.$$

(iii) P is a direct summand of a free R-module

(iv) The covariant functor $M \mapsto \mathrm{Hom}_R(P, M)$ from the category of R-modules to the category of abelian groups is an exact functor.

1.2 Projective resolutions

From these equivalent definitions we can establish some foundational properties of projective resolutions.

Definition I. (Resolutions). Let R be a ring. Given an R-module M, an [adjective] resolution of M is an exact sequence of R-modules

$$\longrightarrow P_n \longrightarrow \cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where each term P_i is an [adjective] R-module. In particular, a projective resolution of M is a resolution as above by projective R-modules P_i.

Exercise 2. Explain why every R-module M has a projective resolution (in fact, a free resolution).

The following exercises are the keys to the construction of morphisms of projective resolutions, and chain homotopies of morphisms.

Exercise 3. (a) Suppose the following diagram

$$\begin{tikzcd} & P \arrow[r, "d"] & Q \arrow[d, "f"] & \\ & N \arrow[r, "d_1"'] & M \arrow[r, "d_2"'] & L \end{tikzcd}$$

satisfies

- The bottom row is exact
- $d_2 \circ f \circ d = 0$
- P is projective

Show that there exists an R-module map g making the diagram commute:

$$\begin{tikzcd} P \arrow[r, "d"] \arrow[d, dotted, "\exists\, g"'] & Q \arrow[d, "f"] & \\ N \arrow[r, "d_1"'] & M \arrow[r, "d_2"'] & L \end{tikzcd}$$

Exercise 4. (a) Suppose the following (not necessarily commutative) diagram

$$\begin{tikzcd} & P \arrow[r, "d"] \arrow[d, "f"] & Q \arrow[dl, "h"] \\ L \arrow[r, "d_0"'] & N \arrow[r, "d_1"'] & M \end{tikzcd}$$

satisfies

- The bottom row is exact
- $d_1 \circ h \circ d = d_1 \circ f$
- P is projective

Show that it is possible to find a map k as below such that $d_0 \circ k + h \circ d = f$.

$$\begin{tikzcd} & P \arrow[r, "d"] \arrow[d, "f"] \arrow[dl, dotted, "\exists\, k"'] & Q \arrow[dl, "h"] \\ L \arrow[r, "d_0"'] & N \arrow[r, "d_1"'] & M \end{tikzcd}$$

With these exercises we can prove the following.

Theorem II. (The Fundamental Theorem of Homological Algebra). *Let R be a ring. Let (C,δ) and (C',δ') be chain complexes of R-modules and let r be an integer. Let $f_i : C_i \to C'_i$ be a family of maps for $0 \leq i \leq r$ making the following diagram commute.*

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{r+1}"] & C_r \arrow[r, "\partial_r"] \arrow[d, "f_r"] & C_{r-1} \arrow[r, "\partial_{r-1}"] \arrow[d, "f_{r-1}"] & \dots \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
\dots \arrow[r] & C'_{r+1} \arrow[r, "\delta_{r+1}"] & C'_r \arrow[r, "\delta_r"] & C'_{r-1} \arrow[r, "\delta_{r-1}"] & \dots \arrow[r, "\delta_2"] & C'_1 \arrow[r, "\delta_1"] & C'_0 \arrow[r] & 0
\end{tikzcd}$$

Assume

- *C_i is projective for all $i > r$,*
- *$H_i(C') = 0$ for all $i \geq r$.*

Then the maps f_i extend to a chain map $f : (C,\delta) \to (C',\delta')$,

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \dots \arrow[r, "\partial_2"] & C_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] & C_0 \arrow[r] \arrow[d, "f_0"] & 0 \\
\dots \arrow[r] & C'_{n+1} \arrow[r, "\delta_{n+1}"] & C'_n \arrow[r, "\delta_n"] & C'_{n-1} \arrow[r, "\delta_{n-1}"] & \dots \arrow[r, "\delta_2"] & C'_1 \arrow[r, "\delta_1"] & C'_0 \arrow[r] & 0
\end{tikzcd}$$

and this chain map is unique up to homotopy. In fact, any two extensions are homotopic by a chain homotopy h such that $h_i = 0$ for all $i \leq r$.

Exercise 5. Prove Theorem II.

Exercise 6. Deduce the following consequences of Theorem II, Theorems III and IV.

Theorem III. *Let R be a ring. Let $P_\bullet \to M$ and $P'_\bullet \to M$ be projective resolutions of an R-module M. Then there exists a chain map f as in the following commutative diagram,*

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & P_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & P_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "\partial_1"] & P_0 \arrow[r, "\partial_0"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "id_M"] & 0 \\
\cdots \arrow[r] & P'_{n+1} \arrow[r, "\delta_{n+1}"] & P'_n \arrow[r, "\delta_n"] & P'_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots \arrow[r, "\delta_1"] & P'_0 \arrow[r, "\epsilon"] & M \arrow[r] & 0
\end{tikzcd}$$

This chain map is unique up to homotopy, and is a homotopy equivalence.

Theorem IV. *Let M, N be modules over a ring R, and let $f_0 : M \to N$ be an R-module map. Then for any projective resolutions $P_\bullet \to M$ and $P'_\bullet \to N$ of M and N, there is a chain map (unique up to homotopy) extending the map f_0.*

1.3 Projective resolutions and short exact sequences

Exercise 7. Let R be a ring and

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of R-modules. Let $P_\bullet \to L$ and $P'_\bullet \to N$ be projective resolutions. Show that there is a projective resolution of M by the projective modules $P_n \oplus P'_n$ making the following diagram commute.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_{n+1} \oplus P'_{n+1} \arrow[r] \arrow[d] & P_n \oplus P'_n \arrow[r] \arrow[d] & P_{n-1} \oplus P'_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \oplus P'_0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P'_{n+1} \arrow[r] \arrow[d] & P'_n \arrow[r] \arrow[d] & P'_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P'_0 \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & & 0 & 0 &
\end{tikzcd}$$

The columns of this diagram are split exact.

2 The Tor functor

The next two sections draw on Dummit–Foote [DF, Chapter 17.1].

2.1 The Tor functor and the associated long exact sequence

Definition V. (The Tor functor). Let R be a ring. Let D be a right R-module and let B be a left R-module. Let

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} B \longrightarrow 0$$

be a projective resolution of B by left R-modules. Then we define the groups $\mathrm{Tor}_*^R(D, B)$ to be the homology of the chain complex

$$\cdots \longrightarrow D \otimes_R P_{n+1} \xrightarrow{1\otimes d_{n+1}} D \otimes_R P_n \xrightarrow{1\otimes d_n} D \otimes_R P_{n-1} \xrightarrow{1\otimes d_{n-1}} \cdots \xrightarrow{1\otimes d_1} D \otimes_R P_0 \longrightarrow 0$$

The following exercises establish well-definedness and some basic properties of Tor.

Exercise 8. Verify that the sequence $(D \otimes P_\bullet, 1 \otimes d_\bullet)$ is indeed a chain complex, that is, verify that

$$(1 \otimes d_{n-1}) \circ (1 \otimes d_n) = 0$$

Exercise 9. Verify that the groups $\mathrm{Tor}_*^R(D, B)$ do not depend on the choice of projective resolution of B. *Hint:* See Theorem III.

Exercise 10. Verify that, for fixed D, the assignment $B \mapsto \mathrm{Tor}_*^R(D, B)$ is functorial. *Hint:* See Theorem IV.

Exercise 11. Verify that there is a natural isomorphism $\mathrm{Tor}_0^R(D, B) \cong D \otimes_R B$.

The following exercise gives some practice in compuing Tor groups.

Exercise 12. (a) Verify that

$$0 \longrightarrow \mathbb{Z} \xrightarrow{m} \mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 0$$

is a projective resolution of the $\mathbb{Z}$-module $\mathbb{Z}/m\mathbb{Z}$.

(b) Let D be a $\mathbb{Z}$-module. Verify that

$$\mathrm{Tor}_0^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong D \otimes \mathbb{Z}/m\mathbb{Z} \cong D/mD$$
$$\mathrm{Tor}_1^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) \cong \{d \in D \mid md = 0\}, \text{ the subgroup annihilated by } m,$$
$$\mathrm{Tor}_n^{\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z}) = 0 \text{ for all } n \geq 2.$$

Exercise 13. Let D be a $\mathbb{Z}/m\mathbb{Z}$-module (and, in particular, a $\mathbb{Z}$-module). What is $\mathrm{Tor}_*^{\mathbb{Z}/m\mathbb{Z}}(D, \mathbb{Z}/m\mathbb{Z})$? Conclude that Tor groups depend on the ring R.

The Tor functors measure, in a sense, the failure of the tensor product to be exact, as we see in the following theorem.

Theorem VI. *Let R be a ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of left R-modules. Then there is a long exact sequence

$$\cdots \to \mathrm{Tor}_{i+1}^R(D,N) \to \mathrm{Tor}_i^R(D,L) \to \mathrm{Tor}_i^R(D,M) \to \mathrm{Tor}_i^R(D,N) \to \mathrm{Tor}_{i-1}^R(D,L) \to \cdots$$

$$\cdots \longrightarrow \mathrm{Tor}_1^R(D,N) \longrightarrow D \otimes_R L \longrightarrow D \otimes_R M \longrightarrow D \otimes_R N \longrightarrow 0$$

Exercise 14. Prove Theorem VI. See Exercise 7.

2.2 Flat modules

Definition VII. (Flat modules). Let R be a ring. A left R-module D is *flat* if the functor $B \mapsto D \otimes_R B$ is an exact functor. Similarly, a right R-module D is *flat* if the functor $B \mapsto B \otimes_R D$ is exact.

Exercise 15. Show that a direct summand of a flat module is flat.

Exercise 16. Show that projective modules are flat.

Exercise 17. Suppose that A and B are flat R-modules. Show that $A \otimes_R B$ is flat.

Exercise 18. Let G be a group. Show that $\mathbb{Q}[G]$ is a flat $\mathbb{Z}[G]$-module.

Exercise 19. Let D be a (left) module over a ring R. Show that the following are equivalent.

(i) D is flat.

(ii) $\mathrm{Tor}_1^R(M,D) = 0$ for every right R-module M.

(iii) $\mathrm{Tor}_i^R(M,D) = 0$ for every right R-module M and all $i \geq 1$.

2.3 Modes of computing $\mathrm{Tor}_*^R(D,B)$

The groups $\mathrm{Tor}_*^R(D,B)$ can in fact be calculated using projective resolutions of either D or B.

Theorem VIII. *Let R be a ring. Let D be a right R-module and let B be a left R-module. Let*

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} D \longrightarrow 0$$

be a projective resolution of D by right R-modules. Then $\mathrm{Tor}_*^R(D,B)$ is equal to the homology of the chain complex

$$\cdots \longrightarrow P_{n+1} \otimes_R B \xrightarrow{d_{n+1}\otimes 1} P_n \otimes_R B \xrightarrow{d_n \otimes 1} P_{n-1} \otimes_R B \xrightarrow{d_{n-1}\otimes 1} \cdots \xrightarrow{d_1 \otimes 1} P_0 \otimes_R B \longrightarrow 0$$

Exercise* 20. Let R be a ring. Let D be a right R-module and let B be a left R-module. Let $P_\bullet \to B$ and $Q_\bullet \to D$ be projective resolutions. Use the double complex $P_\bullet \otimes_R Q_\bullet$ to prove Theorem VIII.
Hint: See Rotman [R, Theorem 10.22].

Exercise 21. Let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of right R-modules. State and prove the existence of the long exact sequence associated to the functor $\mathrm{Tor}_*^R(-, B)$, analogous to Theorem VI.

The following theorem states that, to compute $\mathrm{Tor}_*^R(D, B)$, it suffices to take *flat* resolutions $P_\bullet \to B$ or $Q_\bullet \to D$. The terms P_n and Q_n need not be projective.

Exercise* 22. Prove the following theorem.
Hint: See Rotman [R, Corollary 10.23].

Theorem IX. *Let R be a ring. Let D be a right R-module and let B be a left R-module. Then the description of $\mathrm{Tor}_*^R(D, B)$ given in Definition V (respectively, Theorem VIII) holds even if we assume the resolution of B (respectively, D) is merely flat and not necessarily projective.*

2.4 Change of rings for Tor

Theorem X. *Let $\phi : R \to S$ be a ring homomorphism (preserving unit), so every S-module may be viewed as an R-module. Then*

- *Let B be a right S-module and C a left R-module. If S is flat as a R-module, then there are natural isomorphisms*

$$\mathrm{Tor}_p^R(B, C) \cong \mathrm{Tor}_p^S(B, S \otimes_R C) \qquad \text{for all } p \in \mathbb{Z}.$$

- *Let B be a right R-module and C a left S-module. If S is flat as a R-module, then there are natural isomorphisms*

$$\mathrm{Tor}_p^R(B, C) \cong \mathrm{Tor}_p^S(B \otimes_R S, C) \qquad \text{for all } p \in \mathbb{Z}.$$

Exercise* 23. Prove Theorem X. *Hint:* Consider $S \otimes_R P_\bullet$, where $P_\bullet$ is a free resolution of C by R-modules. See Bieri [Bi, Page 1, Section 2.]. See also Baker [Ba, Example 2.2].

Exercise 24. Let G be a group, M a left $\mathbb{Z}[G]$-module, and V a right $\mathbb{Q}[G]$-module. Prove that

$$\mathrm{Tor}_*^{\mathbb{Z}[G]}(M, V) \cong \mathrm{Tor}_*^{\mathbb{Q}[G]}(M \otimes_{\mathbb{Z}} \mathbb{Q}, V)$$

3 The Ext functor

3.1 The Ext functor and the associated long exact sequences

Definition XI. (The Ext functor). Let A and D be modules over a ring R. Let

$$\cdots \longrightarrow P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

be a projective resolution of A. Then we define the groups $\mathrm{Ext}_R^*(A, D)$ to be the homology of the cochain complex

$$0 \longrightarrow \mathrm{Hom}_R(P_0, D) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, D) \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} \mathrm{Hom}_R(P_{n-1}, D) \xrightarrow{d_n^*} \mathrm{Hom}_R(P_n, D) \xrightarrow{d_{n+1}^*} \cdots$$

Concretely,

$$\mathrm{Ext}_R^n(A, D) = \frac{\ker(d_{n+1}^*)}{\mathrm{im}(d_n^*)}$$

Exercise 25. Verify that the sequence $(\mathrm{Hom}_R(P_\bullet, D), d_\bullet^*)$ is indeed a chain complex.

Exercise 26. Verify that the groups $\mathrm{Ext}_R^*(A, D)$ do not depend on the choice of projective resolution of A.

Exercise 27. Verify that, for fixed D, the assignment $A \mapsto \mathrm{Ext}_R^*(A, D)$ defines a functor from (left) R-modules to abelian groups.

Exercise 28. Verify that there is a natural isomorphism $\mathrm{Ext}_R^0(A, D) \cong \mathrm{Hom}_R(A, D)$.

Exercise 29. Let D be a $\mathbb{Z}$-module. Verify that

$$\mathrm{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong \{d \in D \mid md = 0\}, \text{ the subgroup annihilated by } m,$$
$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) \cong D/mD$$
$$\mathrm{Ext}^n_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = 0 \text{ for all } n \geq 2.$$

As with Tor, the Ext functor in a sense measures the failure of the Hom functor to be exact. It determines the following two long exact sequences.

Theorem XII. *Let R be a ring and*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

a short exact sequence of R-modules. Then there is a long exact sequence

$$0 \longrightarrow Hom_R(N, D) \to Hom_R(M, D) \to Hom_R(L, D) \to \mathrm{Ext}^1_R(N, D) \longrightarrow \cdots$$
$$\cdots \to \mathrm{Ext}^{i-1}_R(L, D) \longrightarrow \mathrm{Ext}^i_R(N, D) \longrightarrow \mathrm{Ext}^i_R(M, D) \to \mathrm{Ext}^i_R(L, D) \to \mathrm{Ext}^{i+1}_R(N, D) \to \cdots$$

and a long exact sequence

$$0 \longrightarrow Hom_R(D, L) \to Hom_R(D, M) \to Hom_R(D, N) \to \mathrm{Ext}^1_R(D, L) \longrightarrow \cdots$$
$$\cdots \to \mathrm{Ext}^{i-1}_R(D, N) \longrightarrow \mathrm{Ext}^i_R(D, L) \longrightarrow \mathrm{Ext}^i_R(D, M) \to \mathrm{Ext}^i_R(D, N) \to \mathrm{Ext}^{i+1}_R(D, L) \to \cdots$$

Exercise 30. Prove Theorem XII. *Hint:* For the first long exact sequence, use Exercise 7. For the second, apply the functors $\mathrm{Hom}_R(-, L)$, $\mathrm{Hom}_R(-, M)$, and $\mathrm{Hom}_R(-, N)$ to a projective resolution for D.

3.2 Injective modules

Definition XIII. (Injective modules). Let R be a ring. A left R-module D is *injective* if the contravariant functor $B \mapsto \mathrm{Hom}_R(B, D)$ is an exact functor from left R-modules to abelian groups.

Exercise 31. Show that $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module. Conclude in particular that projective modules need not be injective.

Exercise 32. Let D be a left module over a ring R. Show that the following are equivalent.

(i) D is injective, that is, $B \mapsto \mathrm{Hom}_R(B, D)$ is an exact functor.

(ii) For any R-modules M, N, any short exact sequence of the following form is split

$$0 \longrightarrow D \longrightarrow M \longrightarrow N \longrightarrow 0.$$

(iii) If D is a submodule of an R-module M, then D has a direct complement in M, that is, there is some $L \subseteq M$ so that M is the internal direct sum $M = D \oplus L$.

(iv) If $f : L \to M$ is an injective map of R-modules, and $g : L \to D$ is any map of R-modules, then there exists a (not necessarily unique) extension of g to M making the following diagram commute.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[dr, "g"'] & M \arrow[d, dashed, "\exists h"] \\
& & D
\end{tikzcd}$$

(v) $\mathrm{Ext}^1_R(M, D) = 0$ for every right R-module M.

(vi) $\mathrm{Ext}^i_R(M, D) = 0$ for every right R-module M and all $i \geq 1$.

Exercise* 33. Prove that every R-module M has an injective coresolution

$$0 \longrightarrow M \xrightarrow{\epsilon} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \longrightarrow \cdots$$

Hint: See Rotman [R, Theorem 3.38].

3.3 Modes of computing $\mathrm{Ext}^*_R(A, D)$

As with Tor, we can compute the groups $\mathrm{Ext}^*_R(A, D)$ using a resolution of either variable.

Theorem XIV. *Let A and D be modules over a ring R. Let*

$$0 \longrightarrow D \xrightarrow{\epsilon} I_0 \xrightarrow{d_1} I_1 \xrightarrow{d_2} I_2 \longrightarrow \cdots$$

be an injective coresolution of D. Then the groups $\mathrm{Ext}^*_R(A, D)$ are equal to the homology of the cochain complex

$$0 \longrightarrow Hom_R(A, I_0) \xrightarrow{d_1^*} Hom_R(A, I_1) \xrightarrow{d_2^*} \cdots \xrightarrow{d_{n-1}^*} Hom_R(A, I_{n-1}) \xrightarrow{d_n^*} Hom_R(A, I_n) \xrightarrow{d_{n+1}^*} \cdots$$

Exercise* 34. Prove Theorem XIV. *Hint:* See Rotman [R, Theorem 6.67].

3.4 Change of rings for Ext

Theorem XV. *Let $\phi : R \to S$ be a ring homomorphism (preserving unit), so every S-module may be viewed as an R-module.*

- *Let A be a left S-module and C a left R-module. If S is flat as an R-module, then there are natural isomorphisms*

$$\mathrm{Ext}^p_R(C, A) \cong \mathrm{Ext}^p_S(S \otimes_R C, A) \qquad \text{for all } p \in \mathbb{Z}.$$

- *Let A be a left R-module and C a left S-module. If S is projective as an R-module, then there are natural isomorphisms*

$$\mathrm{Ext}^p_R(C, A) \cong \mathrm{Ext}^p_S(C, Hom_R(S, A)) \qquad \text{for all } p \in \mathbb{Z}.$$

Exercise* 35. Prove Theorem XV. *Hint:* Consider $S \otimes_R P_\bullet$, where $P_\bullet$ is a free resolution of C by R-modules. Then, consider $\mathrm{Hom}_R(S, I_\bullet)$, where $A \to I_\bullet$ is an injective coresolution of A by R-modules. See Bieri [Bi, Page 1, Section 2.]. See also Baker [Ba, Example 2.4].

Exercise 36. Let G be a group, M a right $\mathbb{Z}[G]$-module, and V a right $\mathbb{Q}[G]$-module. Prove that

$$\mathrm{Ext}^*_{\mathbb{Z}[G]}(M, V) \cong \mathrm{Ext}^*_{\mathbb{Q}[G]}(M \otimes_{\mathbb{Z}} \mathbb{Q}, V)$$

4 Group (co)homology

4.1 The definition of group homology

Definition XVI. (Group homology). Let G be a (discrete) group. Then the *homology of G* is defined to be

$$H_*(G) = \mathrm{Tor}^{\mathbb{Z}[G]}_*(\mathbb{Z}, \mathbb{Z}).$$

More generally, if M is a $\mathbb{Z}[G]$-module, then we define the *homology of G with coefficients in M* to be

$$H_*(G; M) = \mathrm{Tor}^{\mathbb{Z}[G]}_*(\mathbb{Z}, M).$$

Observe that $H_*(G)$ is equal to $H_*(G; \mathbb{Z})$, where $\mathbb{Z}$ is the trivial $\mathbb{Z}[G]$-module.

Using the results on the Tor functor, we can therefore compute $H_*(G)$ in the following ways.

(1) Take a projective (or, more generally, flat) resolution of $\mathbb{Z}$ by right $\mathbb{Z}[G]$-modules,

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

Delete the term $P_{-1} = \mathbb{Z}$ and apply the functor $- \otimes_{\mathbb{Z}[G]} M$. Then $H_*(G; M)$ is the homology of the complex

$$\cdots \longrightarrow P_{n+1} \otimes_{\mathbb{Z}[G]} M \longrightarrow P_n \otimes_{\mathbb{Z}[G]} M \longrightarrow P_{n-1} \otimes_{\mathbb{Z}[G]} M \longrightarrow \cdots \longrightarrow P_0 \otimes_{\mathbb{Z}[G]} M \longrightarrow 0$$

(2) Take a projective (or, more generally, flat) resolution of M by left $\mathbb{Z}[G]$-modules,

$$\cdots \longrightarrow P_{n+1'} \longrightarrow P'_n \longrightarrow P'_{n-1} \longrightarrow \cdots \longrightarrow P'_0 \longrightarrow M \longrightarrow 0$$

Delete the term $P'_{-1} = M$ and apply the functor $\mathbb{Z} \otimes_{\mathbb{Z}[G]} -$. Then $H_*(G; M)$ is the homology of the complex

$$\cdots \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_{n+1} \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_n \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_{n-1} \longrightarrow \cdots \longrightarrow \mathbb{Z} \otimes_{\mathbb{Z}[G]} P'_0 \longrightarrow 0$$

Exercise 37. (a) Let R be a ring. Prove that

$$\operatorname{Tor}_*^{R[G]}(R, B) \cong \operatorname{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, B)$$

for all right $R[G]$-modules B. Conclude that

$$H_*(G; B) \cong \operatorname{Tor}_*^{R[G]}(R, B).$$

In particular, if V is a $\mathbb{Q}[G]$-module, then

$$H^*(G; V) \cong \operatorname{Tor}_*^{\mathbb{Q}[G]}(\mathbb{Q}, V).$$

Hint: Let $F_\bullet \to \mathbb{Z}$ be a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules. First verify that

$$R \otimes_{\mathbb{Z}} F_\bullet \to R \otimes_{\mathbb{Z}} \mathbb{Z} \cong R$$

is still exact.

(b) Describe the two methods of computing the groups $H_*(G; B)$ using $R[G]$-modules (in the spirit of the steps above).

Definition XVII. (Invariants; Coinvariants). Let G be a group and M a left $\mathbb{Z}[G]$-module. The group of *invariants* of M, denoted M^G, is the submodule of M

$$M^G = \{m \in M \mid gm = m \text{ for all } g \in G\}.$$

The group of *coinvariants* of M, denoted M_G, is defined to be the quotient of M

$$M_G = M / \langle gm - m \mid g \in G, m \in M \rangle.$$

The group M^G is the largest submodule of M with trivial G action, and M_G is the largest quotient of M on with trivial G action.

Exercise 38. (a) Let M be a left $\mathbb{Z}[G]$-module. Show that $\mathbb{Z} \otimes_{\mathbb{Z}[G]} M = M_G$.

(b) Formulate the definition of coinvariants for right $\mathbb{Z}[G]$-modules, and prove the analogous result for the functor $- \otimes_{\mathbb{Z}[G]} \mathbb{Z}$.

From this exercise, we can view $H_*(G; M)$ as the homology of the complex

$$\cdots \longrightarrow (P'_{n+1})_G \longrightarrow (P'_n)_G \longrightarrow (P'_{n-1})_G \longrightarrow \cdots \longrightarrow (P'_0)_G \longrightarrow 0$$

arising from a flat resolution $P'_\bullet \to M$. In particular, if we can construct a flat resolution of M such that $(P'_n)_G = 0$ for some n, we can deduce that $H_n(G; M) = 0$.

4.2 The standard resolution

Exercise 39. (The standard resolution). Let C_* be the chain complex defined as follows. The group C_n is the free abelian group

$$C_n = \mathbb{Z}\langle (g_0, g_1, g_2, \ldots, g_n) \mid g_i \in G \rangle$$

with diagonal G-action

$$g \cdot (g_0, g_1, g_2, \ldots, g_n) = (gg_0, gg_1, gg_2, \ldots, gg_n).$$

The differential is

$$\begin{aligned} d_n : C_n &\longrightarrow C_{n-1} \\ (g_0, \ldots, g_n) &\longmapsto \sum_i (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n) \end{aligned}$$

and augmentation $\epsilon : C_0 \to \mathbb{Z}$ defined by $\epsilon(g_0) = 1$ for all $g_0 \in G$.

Verify that C_n is a free $\mathbb{Z}[G]$-module, and d_n is a differential.

(b) Show that, if G is finite, then C_* are the simplicial chains on a simplex with vertex set G. Deduce that C_* is a free resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules.

(c) Show that, for any G, the map

$$\begin{aligned} h : C_n &\longrightarrow C_{n+1} \\ (g_0, g_1, \ldots, g_n) &\longmapsto (1, g_0, g_1, \ldots, g_n) \end{aligned}$$

is a contracting chain homotopy. Deduce that C_* is a free resolution of $\mathbb{Z}$ by free $\mathbb{Z}[G]$-modules. (Note that h is not a map of $\mathbb{Z}[G]$-modules – why is this acceptable?)

(d) To use this resolution to compute $H_*(G)$, we pass to G-coinvariants. Verify that the orbits of $(n+1)$-tuples are uniquely represented by tuples of the form $(1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n)$.

(e) It is standard to write $[g_1|g_2|\cdots|g_n]$ for $(1, g_1, g_1g_2, \ldots, g_1g_2\cdots g_n)$. Describe the groups $(C_n)_G$ the maps induced by the differentials d_n in this new "bar" notation.

Exercise 40. Let $\overline{C}_*$ denote the quotient of the standard resolution C_* by the subcomplex spanned by tuples $(g_0, g_1, \ldots, g_n)$ where $g_i = g_{i+1}$ for some i. Verify that $\overline{C}_*$ is still a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules (called the *normalized standard resolution*) and describe the chain complex of coinvariants.
Hint: Consider the map indcued by h.

Exercise 41. Compute $H_*(G)$ when G is a finite cyclic group.

4.3 The relationship to $K(G,1)$ spaces

Definition XVIII. Let G be a group. A *G-complex* is a CW complex X with an action of G that permutes the cells. The complex X is a *free* G-complex if G freely permutes the cells.

The following exercise shows that the group homology of a group G is equal to the homology of a $K(G,1)$-space, a connected CW complex with fundamental group G and contractible universal cover. Such a CW complex always exists and is unique up to homotopy.

Exercise 42.

(a) Verify that if X is a (free) G-complex, then its cellular chain complex $C_*(X)$ is a complex of (free) $\mathbb{Z}[G]$-modules.

(b) Let X be a connected CW complex with $\pi_1(X) = G$. Explain why its universal cover $p : \tilde{X} \to X$ inherits the structure of a free G-complex, such that G acts transitively on the cells $p^{-1}(\sigma)$ in the preimage of a cell $\sigma \in X$.

(c) Show that $C_*(X) \cong C_*(\tilde{X})_G$.

(d) Let X be a contractible, free G-complex. Show that $C_*(X)$ is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules.

(e) Suppose that X is a connected CW complex with a contractible universal cover. Show that
$$H_*(X) = H_*(G)$$
where $H_*(X)$ is the cellular homology of X, and $H_*(G)$ is the group homology of its fundamental group $G = \pi_1(X)$.

Exercise 43. Describe $K(G,1)$ spaces for the folllowing groups G: $\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, the free group F_n, and the fundamental group of a closed connected orientable surface of genus at least 1.

4.4 The definition of group cohomology

Definition XIX. (Group cohomology). Let G be a (discrete) group. Then the *cohomology of* G is defined to be
$$H^*(G) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z}).$$
More generally, if M is a $\mathbb{Z}[G]$-module, then we define the *cohomology of G with coefficients in M* to be
$$H^*(G; M) = \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

Exercise 44. Describe the two methods of computing $H^*(G;M)$ in the style of Section 4.1.

Exercise 45. Let G be a group and X a $K(G,1)$-space. Show that
$$H^*(X) \cong H^*(G).$$

Exercise 46. Let R be a ring. Prove that
$$\mathrm{Ext}^*_{R[G]}(R, A) \cong \mathrm{Ext}^*_{\mathbb{Z}[G]}(\mathbb{Z}, A)$$
for all $R[G]$-modules A. Conclude that
$$H^*(G; A) \cong \mathrm{Ext}^*_{R[G]}(R, A),$$
and describe the ramifications for computing the groups $H^*(G; A)$.
In particular, if V is a $\mathbb{Q}[G]$-module, then
$$H^*(G; V) \cong \mathrm{Ext}^*_{\mathbb{Q}[G]}(\mathbb{Q}, V).$$

4.5 Some motivation: characteristic classes

In this section, we describe a major reason to study group cohomology.

$K(G,1)$-spaces are special cases of the *classifying space* BG of a topological space G; when G is discrete then BG is precisely the associated $K(G,1)$ space. More generally, we define BG to be a quotient of a (weakly) contractible space EG by a proper free action of G.

The quotient map $EG \to BG$ is a *principal G-bundle*, and is *universal* in the sense that every principal G-bundle over a CW complex X is isomorphic to a *pullback* of this bundle along some map $X \to BG$. In fact, there is bijection

$$\{\text{principal } G\text{-bundles over } X \text{ up to isomorphism}\} \longleftrightarrow \{\text{maps } X \to BG \text{ up to homotopy}\}$$

The cohomology classes $H^*(BG)$ of BG can therefore be used to define invariants of principal G-bundles over X. Let $\xi \in H^*(BG)$ and let $Y \to X$ be a principal G-bundle. Then we may realize this bundle as the pullback along a map $f : X \to BG$, and we may consider the class $f^*(\xi) \in H^*(X)$. In this manner, for each fixed $\xi \in H^*(BG)$ we can associate to every isomorphism class of principal G-bundles over X a cohomology class of X. These cohomology classes are called *characteristic classes*, and measure in a sense the 'twistedness' of the bundle.

Formally, define $\mathcal{E}_G$ to be a contravariant functor from the category of CW complexes and continuous maps to the category of sets,

$$\mathcal{E}_G(X) = \{\text{isomorphism classes of principal } G\text{-bundles over } X\}$$

sending a continuous map of spaces $f : X \to Y$ to a map of G-bundles defined by the pullback operation. The functor $\mathcal{E}_G$ factors through the homotopy category of spaces. A *characteristic class* c of principal G-bundles is defined to be a natural transformation from $\mathcal{E}_G$ to a cohomology functor H^*, viewed as a functor to the category of sets. The Yoneda lemma then implies that there is a bijection

$$\left\{ \begin{array}{c} \text{characteristic classes of principal } G\text{-bundles} \\ c : \mathcal{E}_G \implies H^*(-;A) \end{array} \right\} \longleftrightarrow H^*(BG;A)$$

and so we can understand the cohomology group $H^*(BG;A)$ as parameterizing characteristic classes of principal G-bundles.

There is an analogous theory of characteristic classes for other classes of G-bundles, such as vector bundles ($G = \mathrm{GL}(V)$), M-bundles for a manifold M ($G = \mathrm{Homeo}(M)$), n-sheeted covering spaces ($G = S_n$, the symmetric group), and more. Characteristic classes play a significant role in algebraic topology, differential geometry, and algebraic geometry.

4.6 Induction and coinduction

Toward our eventual goal of defining and studying the virtual cohomological dimension of a group, we now wish to introduce the background needed to state Shapiro's lemma. This result will be a key tool in relating the cohomology of a group to that of its subgroups.

Definition XX. (Induction and co-induction). Let G be a group and $H \subseteq G$ a subgroup. If M is a $\mathbb{Z}[H]$-module, then we can construct from M an *induced* $\mathbb{Z}[G]$-module $\mathrm{Ind}_H^G M$ by extension of scalars:

$$\mathrm{Ind}_H^G M = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M.$$

and a *coinduced* $\mathbb{Z}[G]$-module by

$$\mathrm{Coind}_H^G M = \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M).$$

Exercise 47. Describe how $\mathbb{Z}[G]$ acts on $\mathrm{Ind}_H^G M$ and $\mathrm{Coind}_H^G M$.

Exercise 48.

(a) Describe $\mathbb{Z}[G]$ as a $\mathbb{Z}[H]$-module.

(b) Show that, as an abelian group,

$$\mathrm{Ind}_H^G M \cong \bigoplus_{\sigma H \in G/H} M$$

and explain how G acts on the right-hand side. (Some authors write the above decomposition as $\mathrm{Ind}_H^G M \cong \bigoplus_{\sigma H \in G/H} \sigma M$ to be more suggestive of this G-action).

(c) Show that, as an abelian group,

$$\mathrm{Coind}_H^G M \cong \prod_{\sigma H \in G/H} M$$

(d) Verify that if the index $[G : H]$ is finite, then there is an isomorphism of $\mathbb{Z}[G]$-modules

$$\mathrm{Ind}_H^G M \cong \mathrm{Coind}_H^G M$$

for any $\mathbb{Z}[H]$-module M.
Hint: First show that the map

$$\phi : M \longrightarrow \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$$
$$\phi(m)(g) = \begin{cases} gm, & g \in H \\ 0, & g \notin H \end{cases}$$

extends to a $\mathbb{Z}[G]$-module map $\mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \to \mathrm{Hom}_{\mathbb{Z}[H]}(\mathbb{Z}[G], M)$.

Exercise 49. Let H be a subgroup of a group G. Suppose that N is a $\mathbb{Z}[G]$ module such that

- As an abelian group $N \cong \bigoplus_{i \in I} M_i$
- G permutes the summands transitively, in the sense that there is a transitive action of G on I and $gM_i = M_{gi}$
- H is the stabilizer of M_{i_0} for some i_0

Show that $N \cong \mathrm{Ind}_H^G M_{i_0}$.

Exercise* 50. Show that induction and coinduction are left and right adjoint functors, respectively, to restriction of scalars.

4.7 Shapiro's Lemma

Theorem XXI. (Shapiro's Lemma). *Let H be a subgroup of a group G and let M be an H-module. Then*

$$H_*(H, M) \cong H_*(G; \mathit{Ind}_H^G M) \qquad \textit{and} \qquad H^*(H, M) \cong H^*(G; \mathit{Coind}_H^G M).$$

Exercise 51. Prove Shapiro's Lemma. *Hint:* If F is a free $\mathbb{Z}[G]$-module, then

$$F \otimes_{\mathbb{Z}[H]} M \cong F \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} M \cong F \otimes_{\mathbb{Z}[G]} (\mathrm{Ind}_H^G M).$$

and

$$\mathrm{Hom}_{\mathbb{Z}[H]}(F, M) \cong \mathrm{Hom}_{\mathbb{Z}[G]}(F, \mathrm{Coind}_H^G M).$$

4.8 The rational cohomology of orbit spaces with finite stabilizers

We saw in Exercise 42 that, if X is a contractible simplicial complex with a free simplicial action of a group G, then $H_*(G) \cong H_*(X/G)$, in fact, $H_*(G; A) \cong H_*(X/G; A)$ for any abelian group A. In this section we will see that, in order to compute $H_*(G; \mathbb{Q})$, we may relax our assumptions on the simplicial G-complex X. It will suffice to assume that G acts simplicially with finite stabilizers.

We start with a more general result. The following lemma is formulated as in Church–Putman [CP, Lemma 3.2].

Lemma XXII. *Let G be a group, let X be a simplicial complex on which G acts simplicially, and let Y be a subcomplex of X which is preserved by the G-action. For some $n \geq 0$, assume that the setwise stabilizer subgroup G_σ is finite for every n-simplex σ of X that is not contained in Y. Then the $\mathbb{Q}[G]$-module $C_n(X, Y; \mathbb{Q})$ of relative simplicial n-chains is flat.*

Exercise 52. Prove Lemma XXII.
Hint: For a simplex σ, define M_σ to be the $\mathbb{Q}[G]$-module generated by σ. First use Exercise 49 to argue that

$$M_\sigma \cong \mathrm{Ind}_{G_\sigma}^{G} \mathbb{Q}_\sigma,$$

for an appropriately defined $\mathbb{Q}[G_\sigma]$-module $\mathbb{Q}_\sigma$ with underlying abelian group $\mathbb{Q}$. See the exercises in Section 2.2.

Exercise 53. Let G be a group, let X be a simplicial complex on which G acts simplicially.

(a) Explain why (possibly after barycentrically subdividing) we may assume that the setwise stabilizer G_σ of any simplex σ fixes σ pointwise.

(b) Explain why the orbit space X/G inherits a simplicial structure.

(c) Prove the following theorem.

Theorem XXIII. *Let G be a group, and let X be a contractible simplicial complex on which G acts simplicially. Assume that the stabilizer subgroup G_σ is finite for every simplex σ of X. Then*

$$H_*(G; \mathbb{Q}) \cong H_*(X/G; \mathbb{Q}).$$

5 (Virtual) cohomological dimension

5.1 Projective dimension and cohomological dimension

Definition XXIV. (Projective dimension). Let R be a ring and M an R-module. The *projective dimension* of M, denoted $pd_R(M)$, is the minimal n such that there exists a projective resolution of length n,

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

Theorem XXV. *Let M be a module over a ring R. The following are equivalent.*

(i) $pd_R(M) \leq n$

(ii) $\mathrm{Ext}^i_R(M, -) = 0$ *for* $i > n$.

(iii) $\mathrm{Ext}^{n+1}_R(M, -) = 0$.

(iv) If

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is an exact sequence with each P_i projective, then K is projective.

Exercise 54. Prove Theorem XXV. *Hint:* See Brown [Br, VIII Theorem 2.1].

Definition XXVI. (Cohomological dimension). Let G be a group. Then the *cohomological dimension* of G, denoted $cd(G)$, is defined to be $pd_{\mathbb{Z}[G]}(\mathbb{Z})$.

Exercise 55. Prove the following equalities.

$$\begin{aligned} cd(G) &= pd_{\mathbb{Z}[G]}(\mathbb{Z}) \\ &= \inf\{n \mid \mathbb{Z} \text{ admits a projective resolution of length } n\} \\ &= \inf\{n \mid H^i(G; -) = 0 \text{ for } i > n\} \\ &= \sup\{n \mid H^n(G; M) \neq 0 \text{ for some } \mathbb{Z}[G]\text{-module } M\}. \end{aligned}$$

Exercise 56. Prove $cd(G) = 0$ if and only if G is trivial.

Exercise 57. Use topology to show that, if G is a free group, then $cd(G) = 1$. (Stallings and Swan proved the converse).

5.2 Serre's theorem on finite index subgroups

The goal of this subsection is the following theorem.

Theorem XXVII. (Serre's Theorem). *If G is a torsion-free group and H is a finite-index subgroup, then $cd(H) = cd(G)$.*

The following exercises explore the relationship between the cohomological dimension of a group and its subgroups.

Exercise 58. Let G be a group and $H \subseteq G$ a subgroup.

(a) Let $P_\bullet \to \mathbb{Z}$ be a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules. Explain why $P_\bullet \to \mathbb{Z}$ can be viewed as a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[H]$-modules.

(b) Deduce that $cd(H) \leq cd(G)$.

(c) Further deduce that the complex $(P_\bullet)_H$ computes $H_*(H)$.

Exercise 59.

(a) Prove that a finite cyclic group C has $cd(C) = \infty$.

(b) Conclude that a group with finite cohomological dimension must be torsion-free.

Exercise 60.

(a) Suppose G is a group with finite cohomological dimension. Show that

$$cd(G) = \sup\{n \mid H^n(G; F) \neq 0 \text{ for some free } \mathbb{Z}[G]\text{-module } F\}.$$

Hint: The functor $H^{cd(G)}(G, -)$ is right exact. Any $\mathbb{Z}[G]$-module M is a quotient of a free $\mathbb{Z}[G]$-module.

(b) Let H be a subgroup of a group G. Assume $[G : H] < \infty$, and that G has finite cohomological dimension. Show that $cd(H) = cd(G)$.
Hint: Exercise 48 (d) and Shapiro's Lemma.

Exercise* 61. Prove Theorem XXVII.
Hint: By Exercise 60 it is enough to show

$$cd(H) < \infty \implies cd(G) < \infty.$$

See Brown [Br, Theorem VIII.3.1].

Exercise* 62. Use the Hochschild–Serre spectral sequence to prove the following statement. If

$$0 \longrightarrow K \longrightarrow G \longrightarrow Q \longrightarrow 0$$

is a short exact sequence of groups, then

$$cd(G) \leq cd(K) + cd(Q).$$

Exercise* 63. Show that for any group G there is a *free* resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules of length equal to $cd(G)$.
Hint: See Brown [Br, Proposition 2.6].

5.3 Virtual notions

Definition XXVIII. We say a group G *virtually* has a property if some finite-index subgroup of G has the property. For example, the group G is *virtually torsion-free* if it contains a torsion-free subgroup of finite index.

Proposition XXIX. *If G is virtually torsion-free, then all torsion-free subgroups of G of finite index have the same cohomological dimension.*

Exercise 64. Use Serre's theorem (Theorem XXVII) to prove Proposition XXIX.
Hint: Show that the intersection of finite-index subgroups is finite index.

Proposition XXIX implies that the following concept is well-defined.

Definition XXX. (Virtual cohomological dimension). Let G be a virtually torsion-free group. Then the *virtual cohomological dimension* of G, denoted $vcd(G)$, is the common cohomological dimension of its torsion-free finite-index subgroups.

Exercise 65. Suppose that G is a virtually torsion-free group. Suppose that G acts simplicially on a contractible simplicial complex X, and that the stabilizer G_σ of any simplex σ is finite.

(a) Let H be a torsion-free subgroup of G. Explain why X/H is a $K(H,1)$-space.

(b) Suppose X has dimension $vcd(G)$. Show that

$$vcd(G) = \sup\{n \mid H^n(G; V \otimes_{\mathbb{Z}} \mathbb{Q}) \neq 0 \text{ for some } G\text{-module } V\}.$$

Hint: Consider Exercise **??** and Lemma XXII.

6 Groups of type FP

6.1 Groups of type FP_n and FP_∞

The following definitions generalize the concepts of finite generation and finite presentability.

Definition XXXI. (Finite type resolutions, FP_n). Let R be a ring. A resolution or partial resolution of R-modules is of *finite type* if each term is finitely generated. An R-module M is of *type* FP_n (for some $n \geq 0$) if it admits a partial projective resolution

$$P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

We say that a group G is of *type* FP_n if the trivial representation $\mathbb{Z}$ is a $\mathbb{Z}[G]$-module of type FP_n.

Observe that an R-module M is of type FP_0 precisely if it is finitely generated, and type FP_1 precisely if it is finitely presented.

Exercise* 66. Fix an R-module M and $n \geq 0$. Prove that the following are equivalent.

(i) M admits a partial resolution

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with F_i free of finite rank.

(ii) M is of type FP_n.

(iii) M is finitely generated, and for every partial resolution

$$P_k \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $k < n$, the kernel $\ker\{P_k \to P_{k-1}\}$ is finitely generated.

Exercise 67. Show that the conditions of the following definition are, in fact, equivalent.

Definition XXXII. (Type FP_∞). An R-module M is of *type* FP_∞ if the following equivalent conditions hold.

(i) M admits a free resolution of finite type

(ii) M admits a projective resolution of finite type

(iii) M is of type FP_n for all integers $n \geq 0$.

A group G is of *type* FP_∞ if the trivial representation $\mathbb{Z}$ is a $\mathbb{Z}[G]$-module of type FP_∞.

Exercise 68. Let G be a group and H a finite-index subgroup. Show that G has type FP_n for some $0 \leq n \leq \infty$ if and only if H does.

6.2 Groups of type FP

Definition XXXIII. (Type FP). A group G is of *type* FP if it admits a *finite* projective resolution, that is, a projective resolution of finite type and finite length.

Exercise 69. Verify that a group G is of type FP if and only if G has finite cohomological dimension and is of type FP_∞. *Hint:* Theorem XXV and Exercise 66.

Note that, under the conditions of Exercise 69, $\mathbb{Z}$ has a finite projective resolution by $\mathbb{Z}[G]$-modules of length $cd(G)$),

$$0 \longrightarrow P_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

We may assume moreover that the modules F_i are free, but we may only assume that P_n is projective.

7 (Virtual) duality groups

7.1 Bieri–Eckmann duality groups

The following concept is a generalization of Poincare duality in the context of group (co)homology.

Definition XXXIV. (Bieri–Eckmann duality groups). A group G of type FP is called a *(Bieri–Eckmann) duality group* if there exists an integer n and a $\mathbb{Z}[G]$-module D such that

$$H^i(G; M) \cong H_{n-i}(G; D \otimes_{\mathbb{Z}} M)$$

for all $\mathbb{Z}[G]$-modules M and all integers i. Here, G acts diagonally on $D \otimes_{\mathbb{Z}} M$.

The following exercises show that D must in fact be the $\mathbb{Z}[G]$-module $H^n(G; \mathbb{Z}[G])$, where $n = cd(G)$.

Exercise 70. (a) Review the definition of the groups $H^*(G; \mathbb{Z}[G])$, and explain why they admit a canonical right $\mathbb{Z}[G]$-module structure.

(b) What is $H_*(G; \mathbb{Z}[G])$ (homology instead of cohomology)?

Exercise* 71. Let G be a group of type FP. Then

$$cd(G) = \max\{n \mid H^n(G; \mathbb{Z}[G]) \neq 0\}.$$

Exercise 72. (a) Show that, for any left $\mathbb{Z}[G]$-module M, there is a map

$$\phi : H^*(G; \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} M \longrightarrow H^*(G; M)$$

defined on the level of cochains by mapping $u \otimes m$ ($u \in \mathrm{Hom}_{\mathbb{Z}[G]}(P_i, M)$), $m \in M$) to the cochain $x \mapsto u(x)m$ in $\mathrm{Hom}_{\mathbb{Z}[G]}(P, M)$.

(b) Suppose that G is of type FP and $n = cd(G)$. Prove

$$\phi : H^n(G; \mathbb{Z}[G]) \otimes_{\mathbb{Z}[G]} M \xrightarrow{\cong} H^n(G; M)$$

is an isomorphism for all $\mathbb{Z}[G]$-modules M.
Hint: View ϕ as a natural transformation of right-exact functors of M. See Brown [Br, Proposition VIII.6.8].

(c) Justify the following restatement of this isomorphism. Let $D = H^n(G; \mathbb{Z}[G])$. Then

$$D \otimes_{\mathbb{Z}[G]} M \cong (D \otimes_{\mathbb{Z}} M)_G \cong H_0(G; D \otimes_{\mathbb{Z}} M)$$

so under the assumptions above

$$H^n(G; M) \cong H_0(G; D \otimes_{\mathbb{Z}} M).$$

Bieri–Eckmann proved the following equivalent characterizations of a duality group.

Theorem XXXV. *Let G bee a group of type FP. The following are equivalent.*

(i) G is a duality group, that is, there exists an integer n and a $\mathbb{Z}[G]$-module D such that

$$H^i(G; M) \cong H_{n-i}(G; D \otimes_{\mathbb{Z}} M)$$

for all $\mathbb{Z}[G]$-modules M and all integers i.

(ii) There is an integer n such that $H^i(G, \mathbb{Z}[G] \otimes_{\mathbb{Z}} A) = 0$ for all $i \neq n$ and all abelian groups A.

(iii) There is an integer n such that $H^i(G, \mathbb{Z}[G]) = 0$ for all $i \neq n$ and $H^n(G, \mathbb{Z}[G])$ is a torsion-free abelian group.

(iv) There are natural isomorphisms

$$H^i(G; -) \cong H_{n-i}(G; D \otimes_{\mathbb{Z}} -)$$

where $n = cd(G)$ and $D = H^n(G, \mathbb{Z}[G])$, which are compatible with the connecting homomorphisms in homology and cohomology associated to a short exact sequence of modules.

Exercise* 73. Prove Theorem XXXV. *Hint:* See Brown [Br, Theorem VIII.10.1].

Exercise 74. Choose a group (such as $\mathbb{Z}$) which has a $K(G, 1)$ space equal to a closed orientable manifold. Reconcile Theorem XXXV with conventional Poincaré duality.

7.2 Duality groups over R

Definition XXXVI. (Duality groups over R). A group G of type FP is called a *duality group over a ring R* if there exists an integer n and a (right) $R[G]$-module D such that

$$H^i(G; M) \cong H_{n-i}(G; D \otimes_R M)$$

for all $R[G]$-modules M and all integers i. Here G acts diagonally on $D \otimes_R M$.

Exercise 75. Show that if G is a duality group (over $\mathbb{Z}$) with dualizing module D, then it is a duality group over R with $n = cd(G)$ and dualizing module $D \otimes_{\mathbb{Z}} R$.

Exercise 76. Let G be a duality group over a ring R, with $cd(G) = n$. Show that its dualizing module is

$$D \cong H^n(G; R[G]).$$

Hint: See Bieri [Bi, p144, Claim (f)].

7.3 Virtual duality groups

Exercise 77. Let G be a torsion-free group and H a finite index subgroup. Show that

- G is a duality group if and only if H is
- If G and H are duality groups they have the same dualizing module (under restriction of scalars from $\mathbb{Z}[G]$ to $\mathbb{Z}[H]$).

Hint: Show that the isomorphism $H^*(H; \mathbb{Z}[H]) \cong H^*(G; \mathbb{Z}[G])$ implied by Shapiro's lemma is in fact an isomorphism of $\mathbb{Z}[H]$-modules.

Definition XXXVII. (Virtual duality group). A group G is a *virtual duality group* if some subgroup of finite index is a duality group.

We will see that, if G is a virtual duality group, then every torsion-free finite-index subgroup is a duality group with dualizing module $H^{cd(G)}(G; \mathbb{Z}[G])$.

Exercise 78. Use Shapiro's lemma to prove the following proposition.

Proposition XXXVIII. *A group G is virtual duality group if and only if the following two conditions are satisfied*

(a) G has a finite-index subgroup of type FP. (The group G is said to be of type VFP.)

(b) There is an integer n such that $H^i(G; \mathbb{Z}[G]) = 0$ for all $i \neq n$ and $H^n(G; \mathbb{Z}[G])$ is a torsion-free abelian group.

In this case, every torsion-free subgroup of finite index is a duality group with dualizing module $H^n(G; \mathbb{Z}[G])$.

Exercise* 79. Let G be a group without R-torsion, and let H be a finite-index subgroup. Show that G is a duality group over R if and only if H is.
Hint: This is Bieri [Bi, Section 9.6, Theorem 9.9].

Exercise 80. Use Exercise 79 to prove the following proposition.

Proposition XXXIX. *Let G be a virtual duality group. Then G is a duality group over $\mathbb{Q}$. In particular, for every i and every $\mathbb{Q}[G]$-module V,*

$$H^{vcd(G)-i}(G;V) \cong H_i(G; D \otimes_{\mathbb{Q}} V)$$

where $D \cong H^{vcd(G)}(G;\mathbb{Q}[G])$ is the common rational dualizing module of G and its finite index subgroups.

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Please send comments & corrections!
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