Lecture 1: The curve complex and duality for the mapping class group

June 28, 2021

1.1

$$S = S^b_{g,n} \quad \begin{cases} g = \text{genus} \\ n = \text{punctures / marked pts} \\ b = \#\text{ of } \partial\text{-comps} \end{cases}$$

punctures also fixed ptwise

$$\mathrm{PMod}(S) := \mathrm{Diff}^+(S, \partial S) / \mathrm{Diff}_0(S, \partial S)$$

$$\mathrm{Teich}(S) := \{(\underline{X}, \underline{\phi}) \mid \phi : \underline{S} \to X\} / \sim$$

$\sim$:

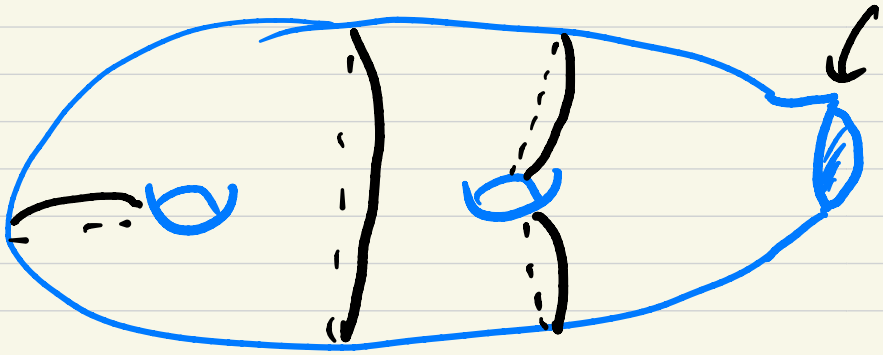
$$\begin{tikzcd} & X_1 \arrow[d, "I"] \\ S \arrow[ur, "\phi_1"] \arrow[r, "\circlearrowright"] \arrow[dr, "\phi_2"'] & X_1 \\ & X_2 \end{tikzcd}$$

htpy

1.2

$$\mathrm{Teich}(S) \cong \mathbb{R}^d$$

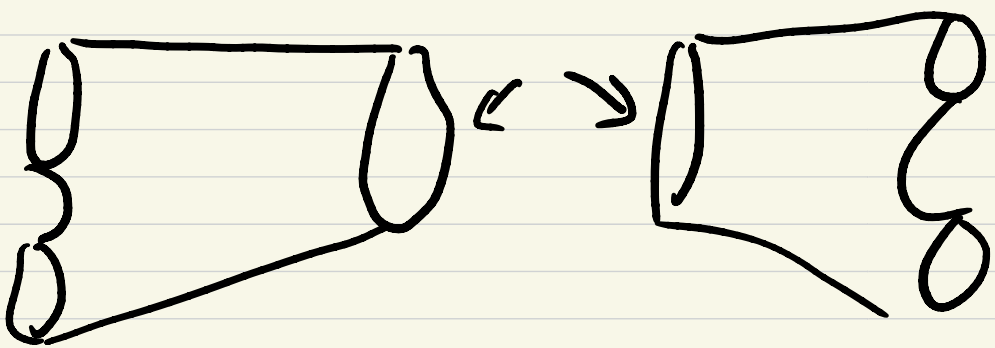
$$d = 6g - 6 + 3b + 2n$$



$$= 3g - 3 + 2b \quad \text{length}$$

$$+ \; 3g - 3 + b \quad \text{twist}$$

$$6g - 6 + 3b$$



$\mathrm{Mod}(S)$ acts: $f \cdot (X, \phi) = (X, \phi \circ f^{-1})$

The moduli space of Riemann surfaces

$$\mathcal{M}(S) := \mathrm{Teich}(S) / \mathrm{PMod}(S)$$

is an orbifold classifying space

$$H^*(\mathrm{PMod}(S); \mathbb{Q}) \cong H^*(\mathcal{M}(S); \mathbb{Q})$$

Analogy w/ $SL_n\mathbb{Q}$:

rational Tits building $\Delta(n, \mathbb{Q})$

abstract simplicial complex of flags of non-trivial, proper subspaces of $\mathbb{Q}^n$

k-simplices $\leftrightarrow$ $(k+1)$ 'connected' spaces

Harer

Thm: (1) $\Delta(n,\mathbb{Q}) \simeq \bigvee_{\infty} (S^{n-2})$

Solomon-Tits

(2) $St(n) := \tilde{H}_{n-2}(\Delta(n,\mathbb{Q});\mathbb{Z})$ is cyclic as an $SL(n,\mathbb{Q})$-module

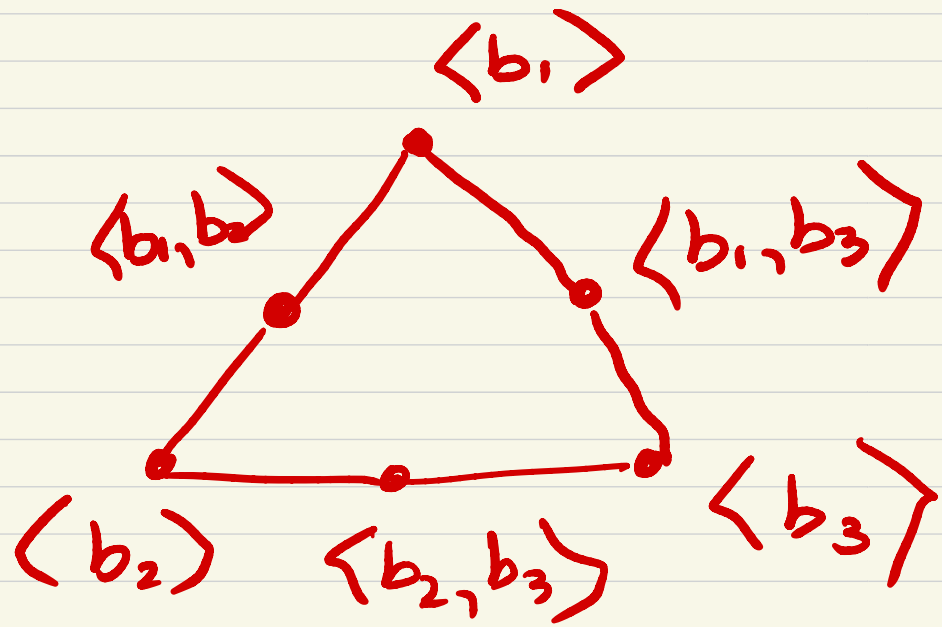
Broadbus

Ash-Rudolph

(3) $St(n)$ " " " $SL(n,\mathbb{Z})$-module

Ex $n=3$

Basis $\{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ for $\mathbb{Q}^3$

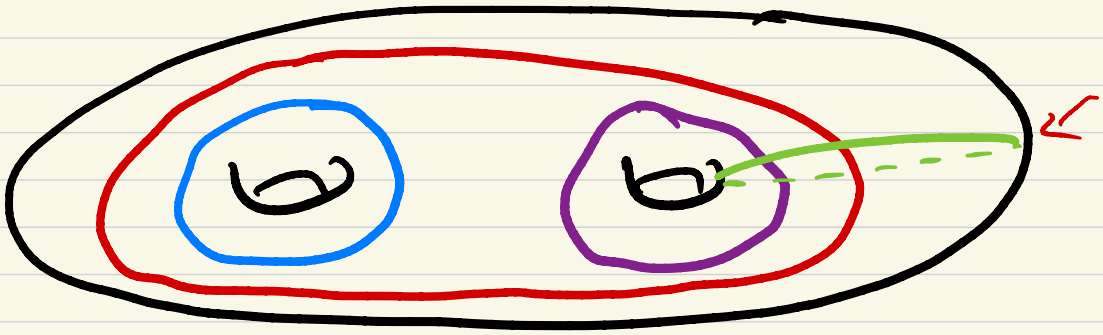


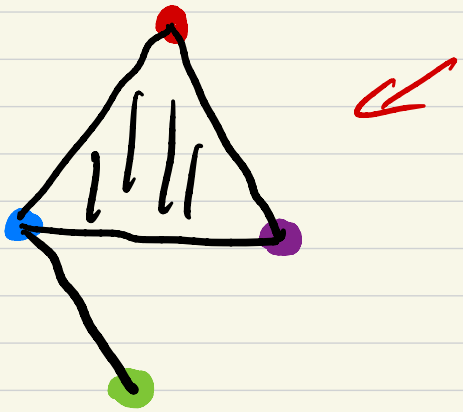
Harvey '77

1.6

Analogue for Mod(S): the curve complex $\mathcal{C}(S)$

K-simplices $\longleftrightarrow$ (K+1) disjoint isotopy classes of scc's (*)





Note: $\mathcal{C}(S_{g,n}^{b}) \cong \mathcal{C}(S_{g,n+b})$

(*) 'curve system'

1.7

Thm (Harer '86)

$$\mathcal{C}(S_{g,n}) \simeq \bigvee_{\infty} S^m$$

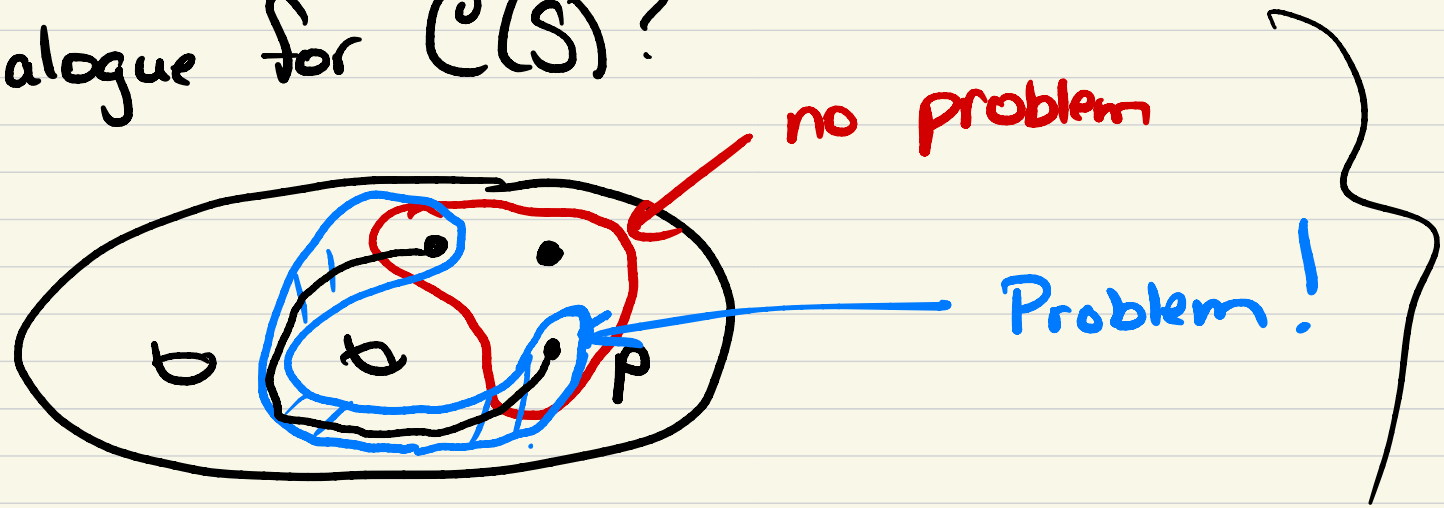
where $m = \begin{cases} n-4 & \text{if } g=0 \\ 2g-2 & \text{if } g \geq 1,\ n=0 \\ 2g-3+n & \text{if } g \geq 1,\ n \neq 0 \end{cases}$

$$\dim(\mathcal{C}(S_{g,n})) = 3g+n-4$$

Birman Exact Sequence

$$1 \to \pi_1(S_{g,n}) \to \mathrm{PMod}(\underline{S_{g,n}}) \xrightarrow{\text{forget}} \mathrm{PMod}(\underline{S_{g,n-1}}) \to 1$$

Analogue for $\mathcal{C}(S)$?



$\mathcal{X}(S_{g,n})$ = sbcx of $\mathcal{C}(S_{g,n})$ spanned by 'good' curves

$\mathcal{A}_p$ = 'bad' curves

$$\mathcal{X}(S_{g,n}) \longrightarrow \mathcal{C}(S_{g,n-1})$$

<u>Prop</u> (Harer) $\mathcal{C}(S_{g,n}) \simeq A_p * \mathcal{X}(S_{g,n})$

$$\simeq A_p * \mathcal{C}(S_{g,n-1})$$

Proof in
B-Broaddus-Putman

Prop (Harer) $\underline{\mathcal{C}(S_{g,n})} \simeq A_p * \mathcal{X}(S_{g,n})$

$$\simeq A_p * \underline{\mathcal{C}(S_{g,n-1})}$$

Exercise: X any simplicial complex

A any discrete set

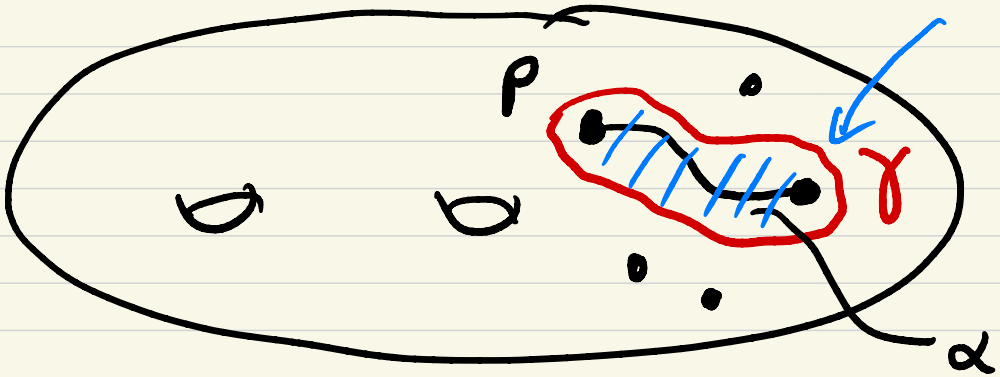
$Y \subseteq A * X$, and $A, X \subseteq Y$ s.t.

$\forall a \in A$, $\mathrm{lk}_Y(a) \hookrightarrow X$ is a htpy eq.

Then $Y \hookrightarrow A * X$ is a htpy eq.

$X = \mathcal{X}(S_{g,n})$, $A = A_p$

1.10



Claim: $\mathrm{lk}_{\mathcal{C}(S_{g,n})}(\gamma) \xrightarrow{\iota} \mathcal{X}(S_{g,n})$ is a htpy eq.

$$\begin{tikzcd} \mathcal{X}(S_{g,n}) \arrow[d, "\pi"] \\ \mathcal{C}(S_{g,n-1}) \end{tikzcd}$$

- $\pi \circ \iota$ is a simpl. isom.

WTS ι_* is surjection

Fix k, simplicial structure

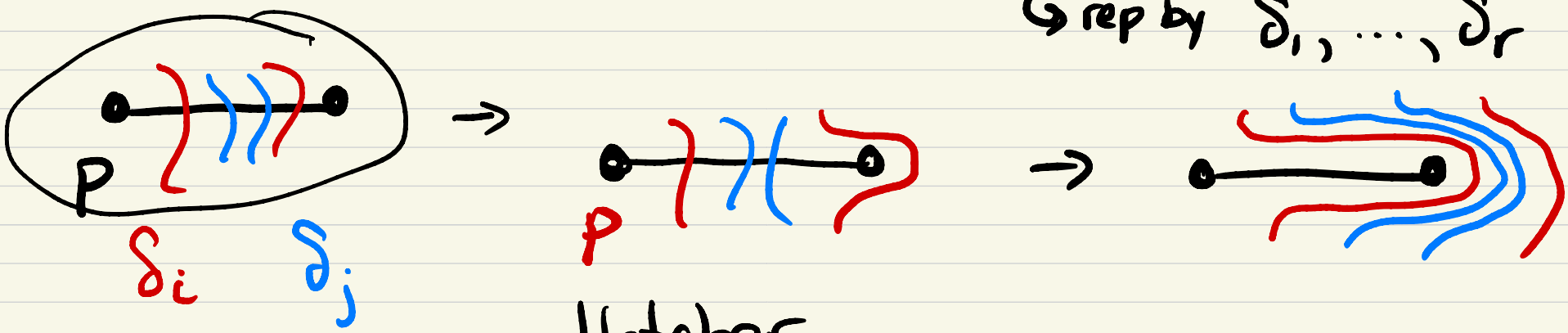
1.11

Consider $\psi : S^k \longrightarrow \mathcal{X}(S_{g,n-1})$.

Homotope ψ s.t. $\psi(S^k) \cap \alpha = \phi$ (α: γ)

– Enough to do this on vertices $v_1, \dots, v_r$ of S^k

↳ rep by $\delta_1, \dots, \delta_r$

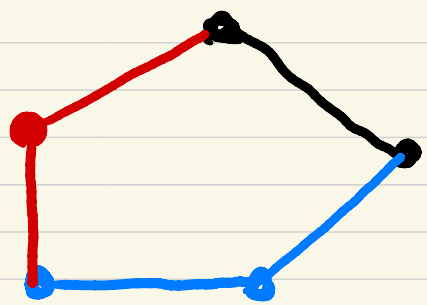


Hatcher

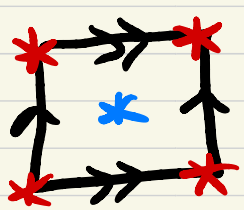
$\mathcal{C}(S)$ is an infinite collection of vertices for

$$S \in \{ S_{0,4}, \boxed{S_{1,0}, S_{1,1}} \}$$

$$\mathcal{C}(S_{0,5}) \simeq \bigvee_{\infty} S^1 \qquad \dim \mathcal{C}(S_{0,n}) = n-4$$



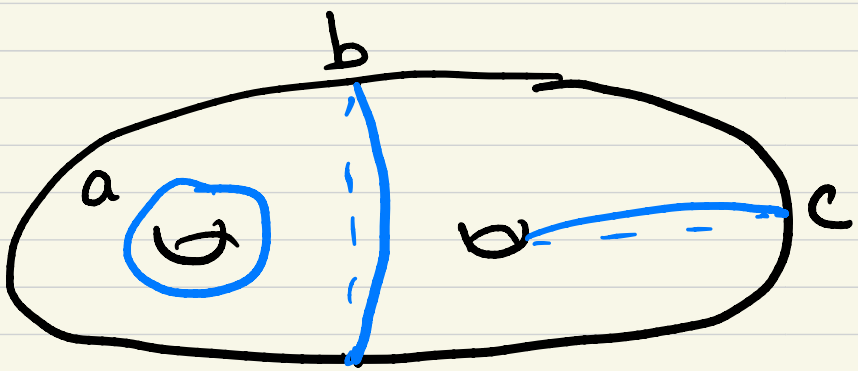
Exercise: $\mathcal{C}(S_{0,5}) \cong \mathcal{C}(S_{1,2})$

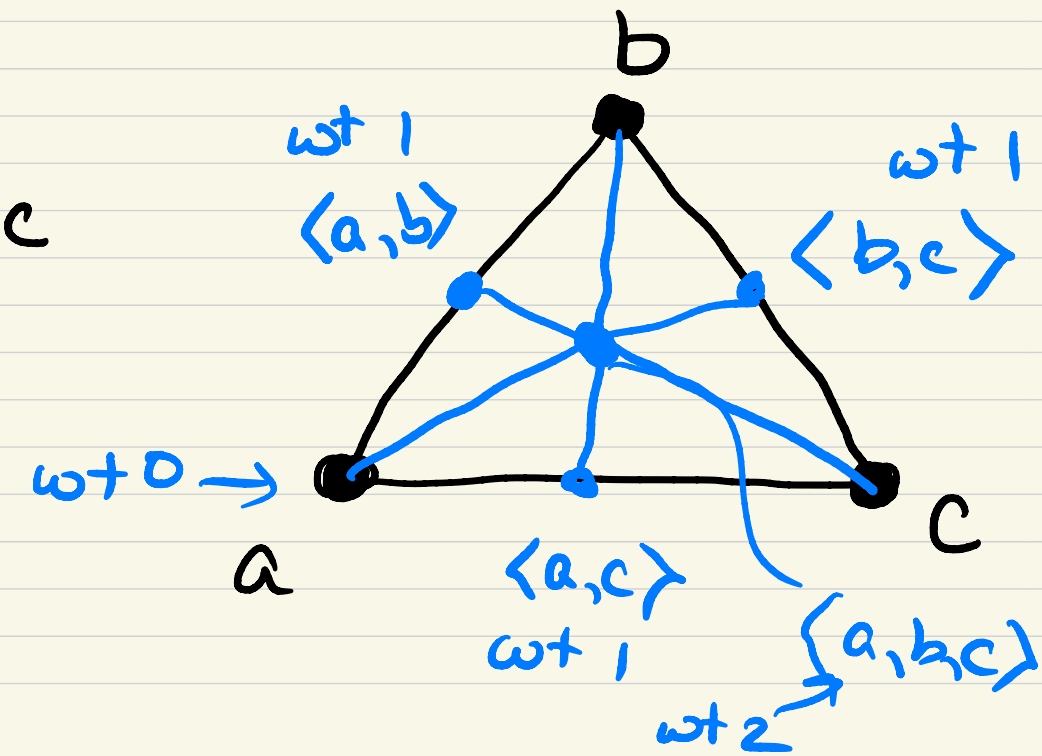


Progress! We have reduced the proof of Harer's theorem to showing:

$$\mathcal{C}(S_{g,0}) \simeq \bigvee S^{2g-2} \quad \text{for } g \geq 2$$

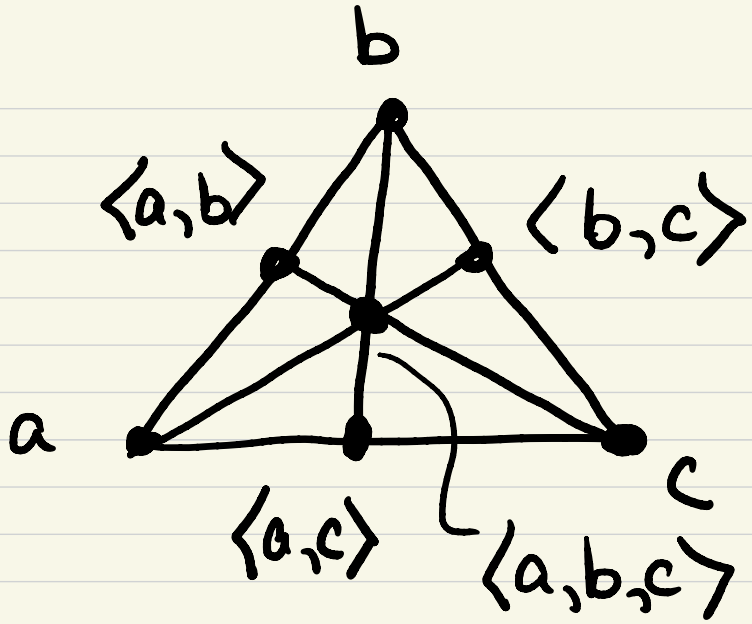
Consider first barycentric subdiv $Z(S)$ of $\mathcal{C}(S)$





weight

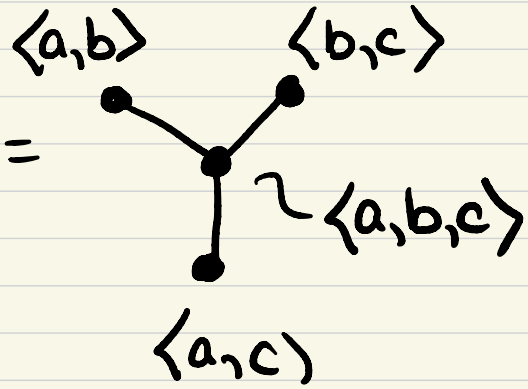
1.14



$Z_K :=$ subcx of Z spanned by vertices of weight $\geq K$

So... $Z_0 = Z$

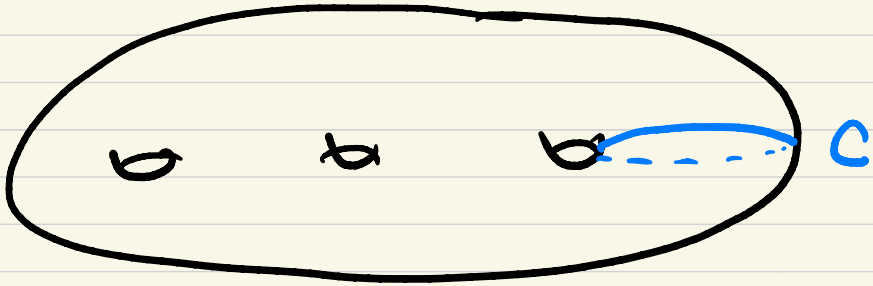
Z_1(above) =



$Z_2 =$ • ⟨a,b,c⟩

1.15

Choose $v \in Z_0 \setminus Z_1$, with representative scc c.



cut S along c, get S_c

$$\mathrm{lk}_{Z_1}(v) = \mathcal{C}(S_c) \cong \mathcal{C}(S_{g-1,2})$$

htpy eq. to Spheres of dim

$$2(g-1) - 3 + 2 = 2g-3$$

now 'add' $v * \mathrm{lk}_{Z_1}(v) \rightsquigarrow$ dim $2g-2$

1.17

Thm (Harer)

$$\mathrm{vcd}\left(\mathrm{PMod}\left(S_{g,n}^{b}\right)\right) = \begin{cases} 2b+n-3 & g=0 \\ 4g-5 & g\geq 2,\ n=b=0 \\ 4g+2b+n-4 & g\geq 1,\ n+b\geq 1 \end{cases}$$

$$\mathrm{htpy\ dim}\ \mathcal{C}\left(S_{g,n}^{b}\right) = \begin{cases} n+b-4 & g=0 \\ 2g-2 & g\geq 1,\ n=b=0 \\ 2g-3+n+b & g\geq 1,\ n+b\geq 1 \end{cases}$$

$$\dim\left(\mathrm{Teich}\left(S_{g,n}^{b}\right)\right) = 6g-6+3b+2n$$

1.18

$X = X^d$ a $K(G,1)$ w/ $\partial X \neq \phi$

$\tilde{X}$ its universal cover

Poincaré duality:

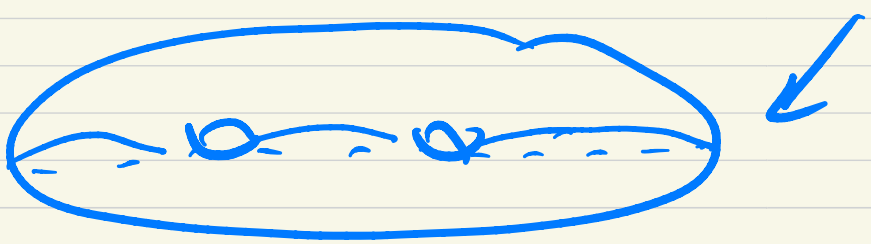
$$H^k(X; \mathbb{Z}G) \cong H_{d-k}(X \bmod \partial X; \mathbb{Z}G)$$
$$\cong H_{d-k}(\tilde{X}, \partial\tilde{X}; \mathbb{Z})$$
$$\cong H_{d-k-1}(\partial\tilde{X}; \mathbb{Z})$$

G a duality gp of dim v

$$\Rightarrow \quad H^k(X; \mathbb{Z}G) \cong H_{d-k-1}(\partial\tilde{X}; \mathbb{Z}) = 0 \quad \forall k \neq v$$

When $k = v$, get:

$$H^v(X; \mathbb{Z}G) \cong H_{d-v-1}(\partial \tilde{X}; \mathbb{Z})$$

$\partial \tilde{X}$ has vanishing integral homology in all dim except $m = d - v - 1$

htpy dim $\mathcal{C}(S)$ — dim Teich(S) — vcd Mod(S)