Lecture 2: The Steinberg module

((of the mapping class group

June 29, 2021

N. Broaddus '12

2.1

Last time: the curve complex $C(S)$

k-simplices $\leftrightarrow$ 'curve systems'

= $(k+1)$ disjoint scc's

The Steinberg module

$$St(S) := \tilde{H}_{2g-2}(\mathcal{C}(S); \mathbb{Z})$$

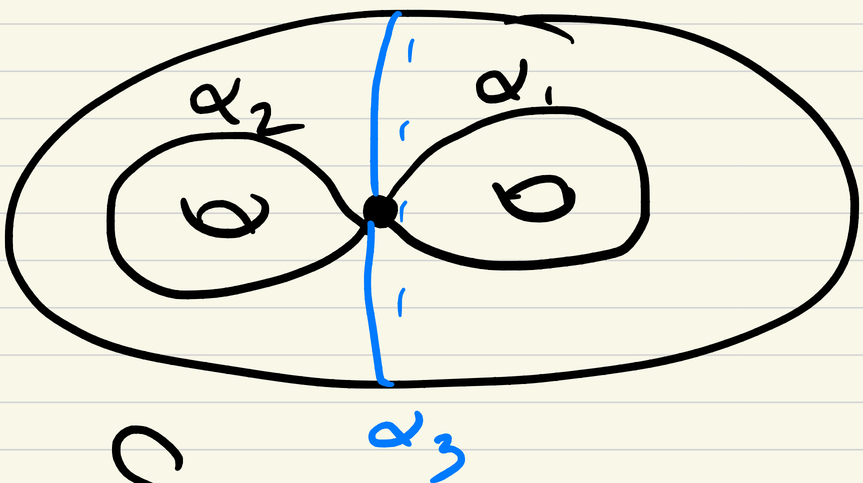
Focus: $S = S_g$ or $S_{g,1}$

Note: $\mathcal{C}(S_g) \cong \mathcal{C}(S_{g,1})$ Harer, Kent-Leininger-Schleimer

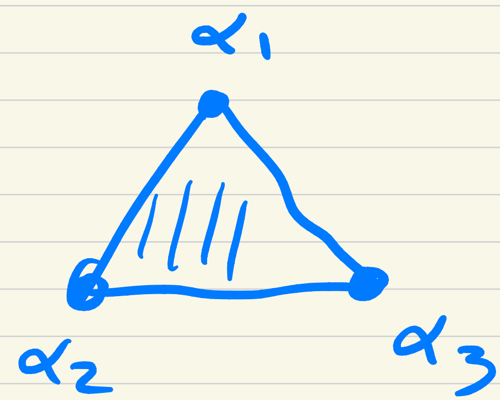
2.2

Variation: the arc complex $\mathcal{A}(S)$ $(n>0)$

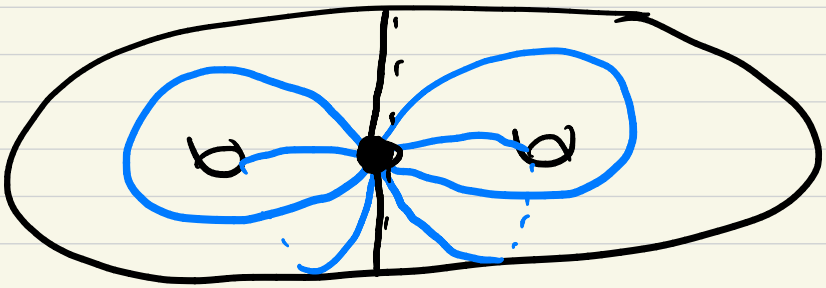
k-simplices $\longleftrightarrow$ 'arc systems'
(k+1) arcs disjoint except maybe at endpoints



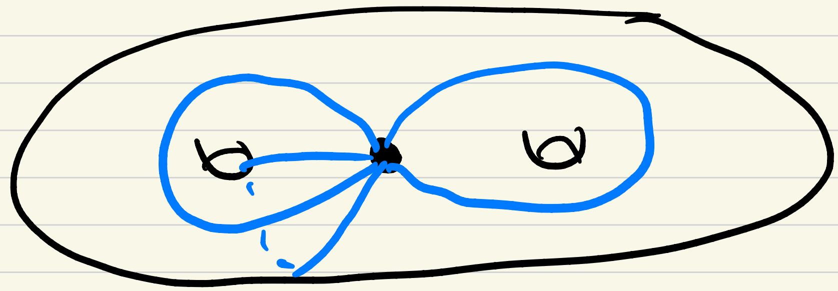
$S_{g,1}$



The arc complex at infinity $\mathcal{A}_\infty$ (2.3)

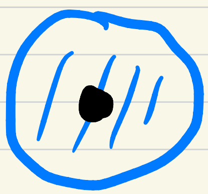


a filling arc system:



non-filling

cuts $S_{g,n}$ into disks:

 or 

k-filling system $\leftrightarrow$ filling system w/ $(2g+k)$ arcs

$\mathcal{A}_\infty :=$ sbcx of $\mathcal{A}$ spanned by non-filling systems

Thm (Mumford, Thurston / Bowditch - Epstein)

$$\underline{\mathrm{Teich}(S)} \underset{\text{'nice'}}{\cong} \underline{\mathcal{A}(S)} \setminus \underbrace{\mathcal{A}_\infty(S)}_{\text{codim } 2}$$

$(X, \phi) \rightsquigarrow$ unique 'horocyclic' quad diff

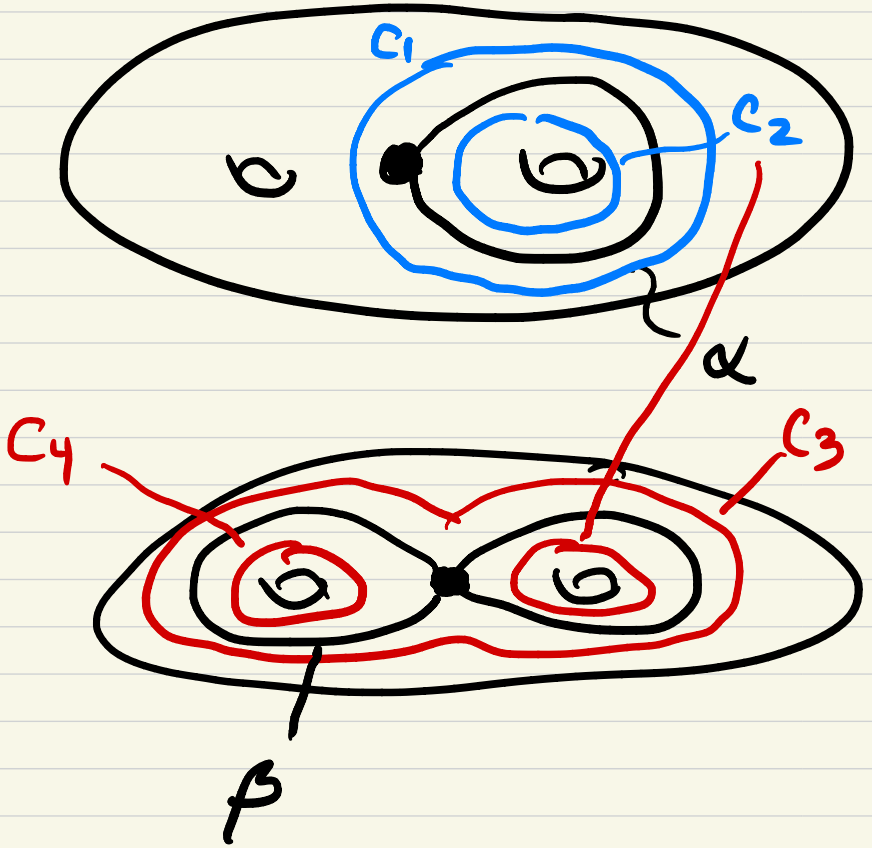
$\rightsquigarrow$ locally

$\longmapsto$ filling arc system

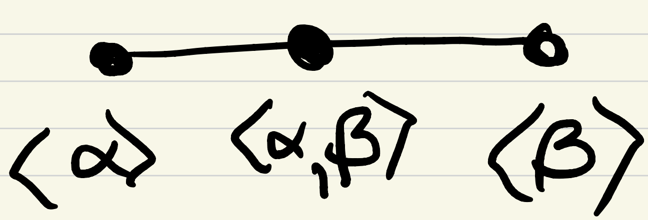
Jenkins, Strebel

2.5

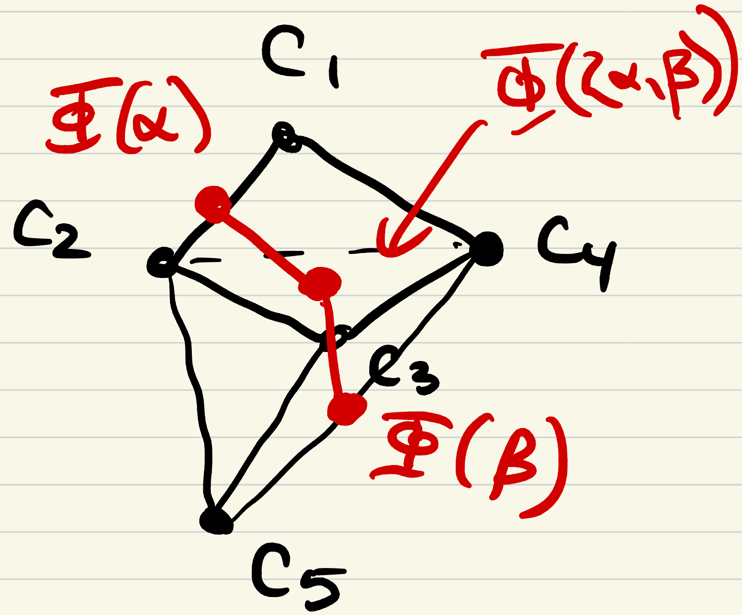
Define $\Phi : A_\infty(S) \xrightarrow{\cong} \dfrac{\mathcal{C}(S)}{Z(S)}$



$\longmapsto \quad \langle c_1, c_2 \rangle$



$\longmapsto$



2.6

Two key facts:

(1) $\mathcal{A}_\infty(S_{g,1}) \simeq \bigvee S^{2g-2}$ (we did this yesterday)

(2) $\mathcal{A}(S_{g,1})$ is contractible (Hatcher flow)

(Both theorems of Harer)

2.7

Long exact sequence for $(\mathcal{A}, \mathcal{A}_\infty)$:

$$\cdots \to \underbrace{H_{k+1}(\mathcal{A};\mathbb{Z})}_{0} \to \underline{H_{k+1}(\mathcal{A}/\mathcal{A}_\infty;\mathbb{Z}) \to \tilde{H}_k(\mathcal{A}_\infty;\mathbb{Z})} \to \underbrace{\tilde{H}_k(\mathcal{A};\mathbb{Z})}_{0} \to \cdots$$

For $k \geq 0$, get an isom

$$St(S_{g,1}) \cong H_{2g-1}(\mathcal{A}/\mathcal{A}_\infty;\mathbb{Z})$$

'Shifted chain complex' for $\mathcal{A}/\mathcal{A}_\infty$:

$$F_k := C_{2g-1+k}(\mathcal{A}/\mathcal{A}_\infty; \mathbb{Z})$$

<u>Prop</u> (Broaddus)

$$0 \to F_{4g-3} \xrightarrow{\partial} \cdots \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \to \mathrm{St}(S_{g,1}) \to 0$$

is a $\mathrm{Mod}(S_{g,1})$-resolution for $\mathrm{St}(S_{g,1})$.

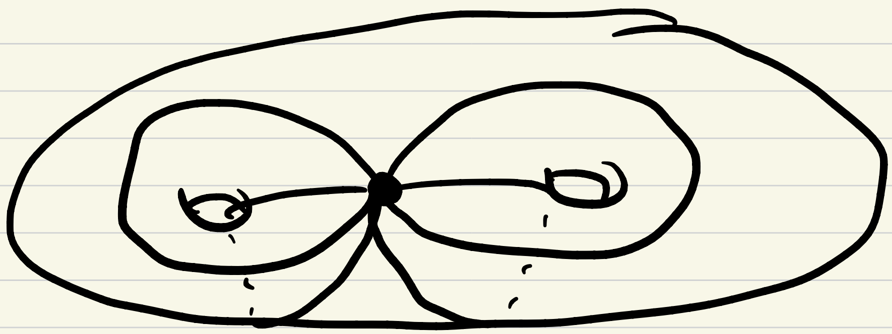
2.9

$$St(S_{g,1}) \cong F_0 / \partial F_1$$

(oriented) 0-filling arc systems

1-filling arcs systems

Ex

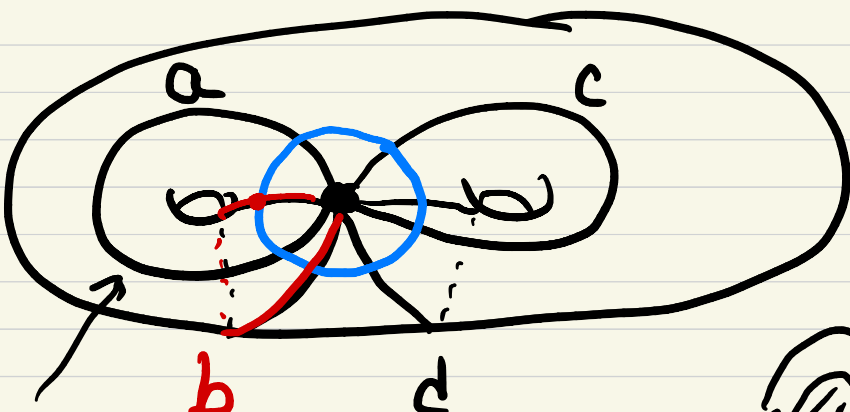


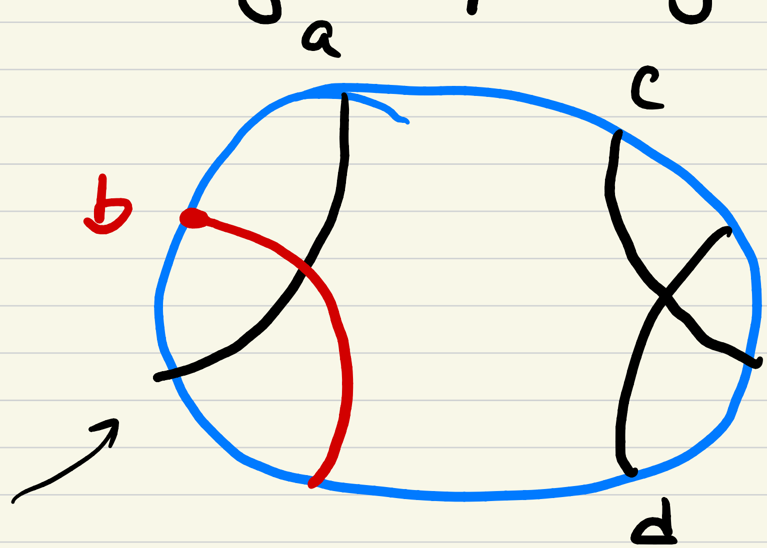
0-filling system $\leftrightarrow$ 2g arcs

$$\partial(\alpha_0, \alpha_1, \cdots, \alpha_{2g}) = \sum_{k=0}^{2g} (-1)^k (\alpha_0, \cdots, \hat{\alpha}_k, \cdots, \alpha_{2g})$$

By def'n, F_0 is generated as a $\mathrm{Mod}(S_{g,1})$-module by distinct topological types of 0-filling arc systems

Enumerate these via chord diagrams / fat graphs





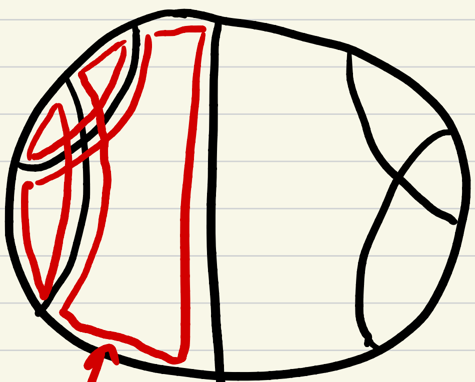
chord diagram

2.11

Topological properties of chord diagrams:

1) n = # of arcs

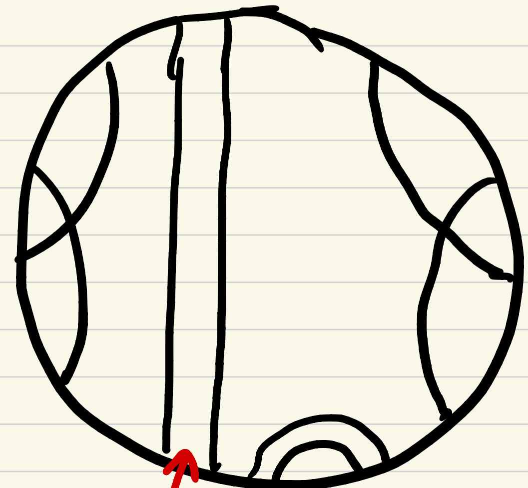
2) β = # of 'boundary components



$\beta = 2$

∂-comp

3) Parallel arcs:



rectangle

$$g = \frac{n + 1 - \beta}{2}$$

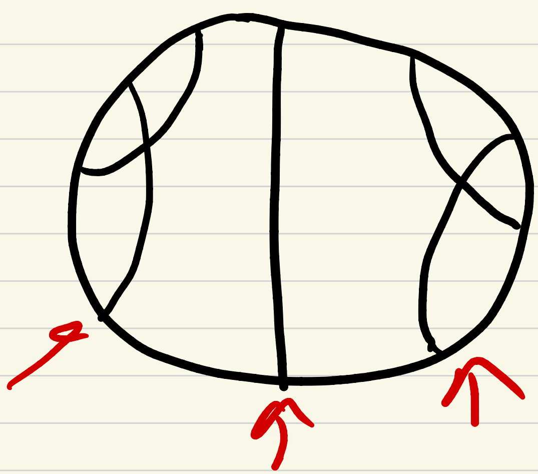
2.12

Topological properties of chord diagrams:

4) connected components:

- partition the set of chords s.t. each pair that crosses is in the same set
- finest such partition gives the connected components

Ex

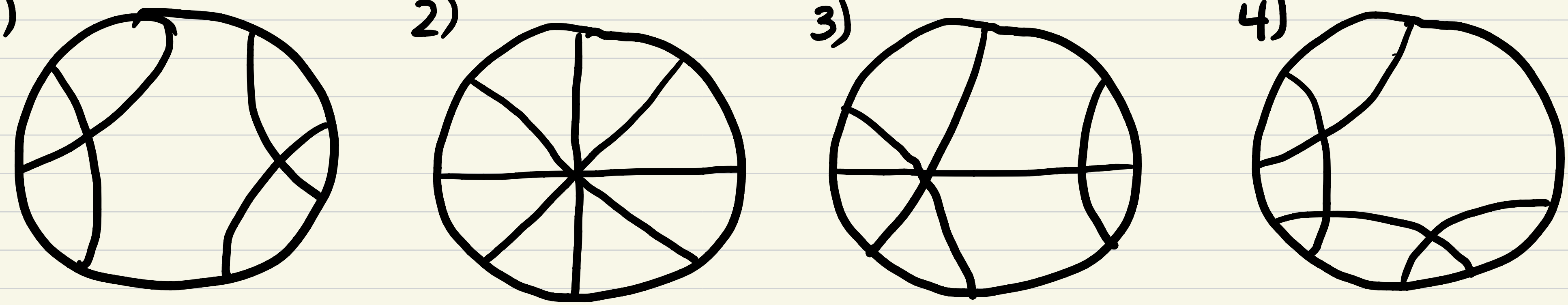


has 3 conn. comps

2.13

$S_{2,1}$ admits 4* distinct topological types of chord diagrams from 0-filling systems

1) 2) 3) 4)



* Always finite

2.14

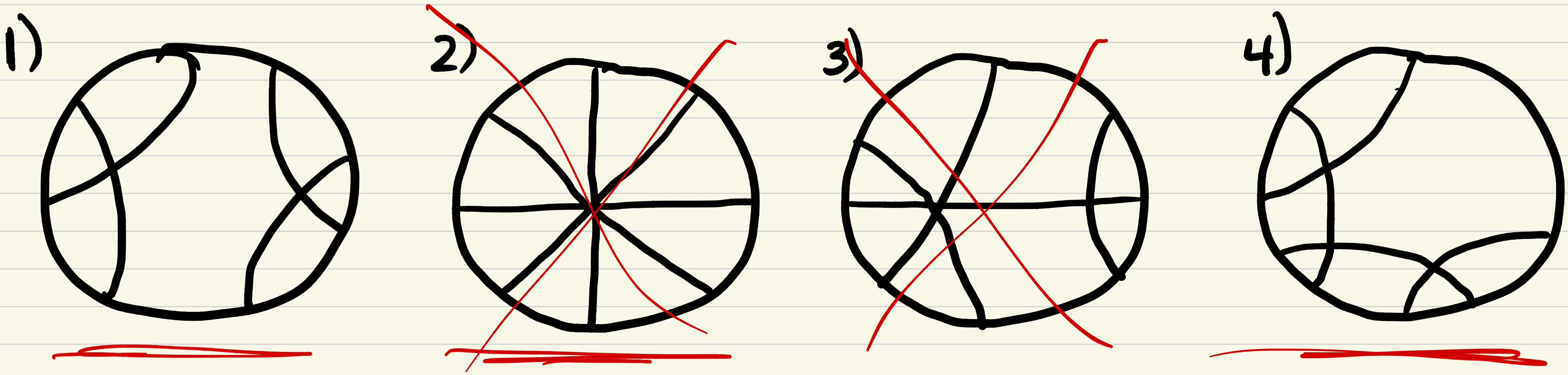
<u>Theorem</u> (Broaddus) $(g \geq 1)$

$\mathrm{St}(S_{g,1})$ is cyclic as a $\mathrm{Mod}(S_{g,1})$-module

2.14

Theorem (Broaddus)

$St(S_{g,1})$ is cyclic as a $Mod(S_{g,1})$-module

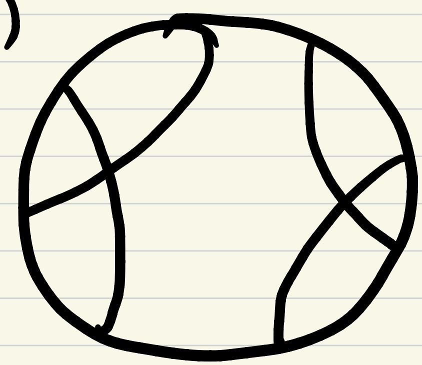


For $S_{2,1}$ which is it?

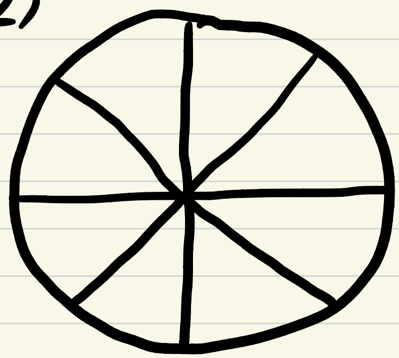
2.15

And the winner is . . .

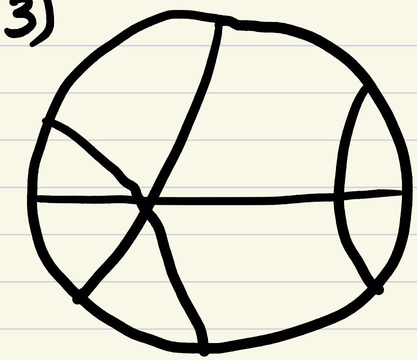
1)



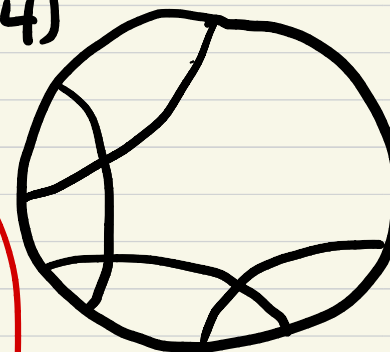
2)



3)

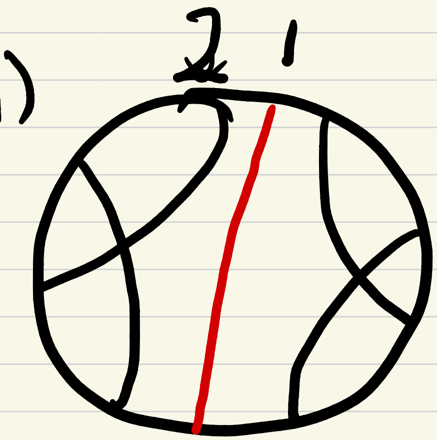


4)



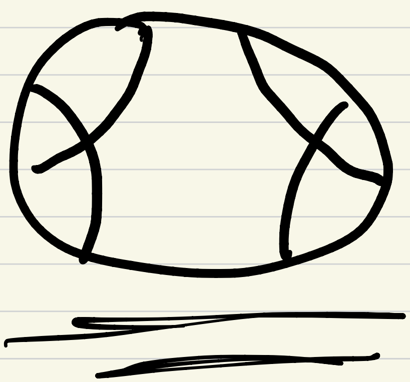
2.16

<u>Proof:</u> Rule out 1)

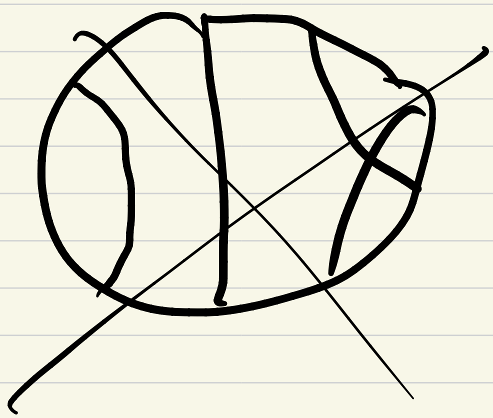


Insert c

Compute its image under ∂



$-$

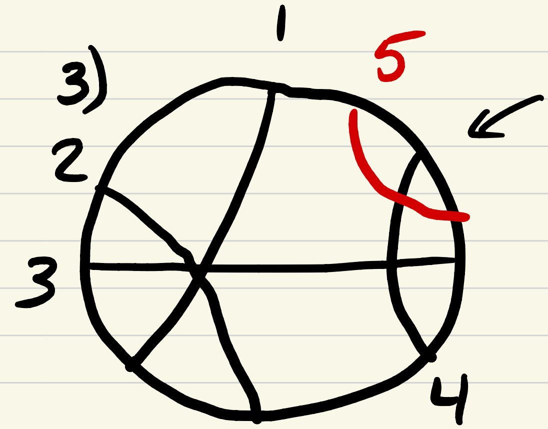


$+ \dots$ parallel arcs

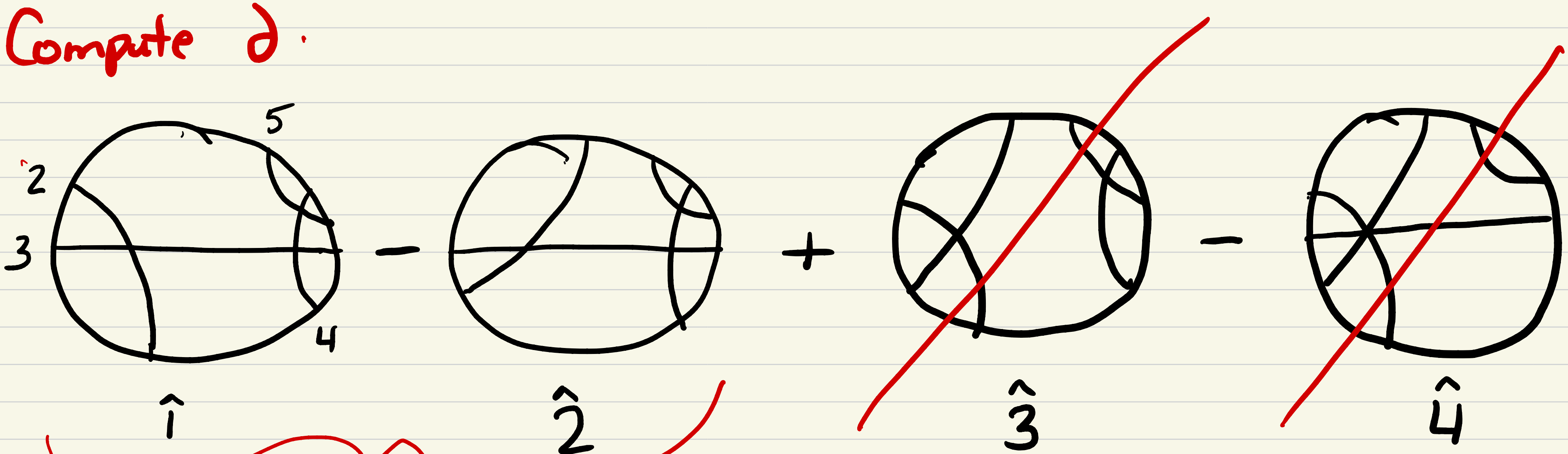
$\in \partial F_1$

2.17

Next look at, say,



Compute ∂:

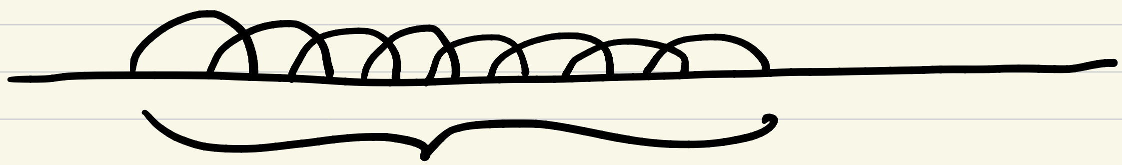


longer 'salient tails'

disconnected so in ∂F_1

+ original

Salience:



n chords

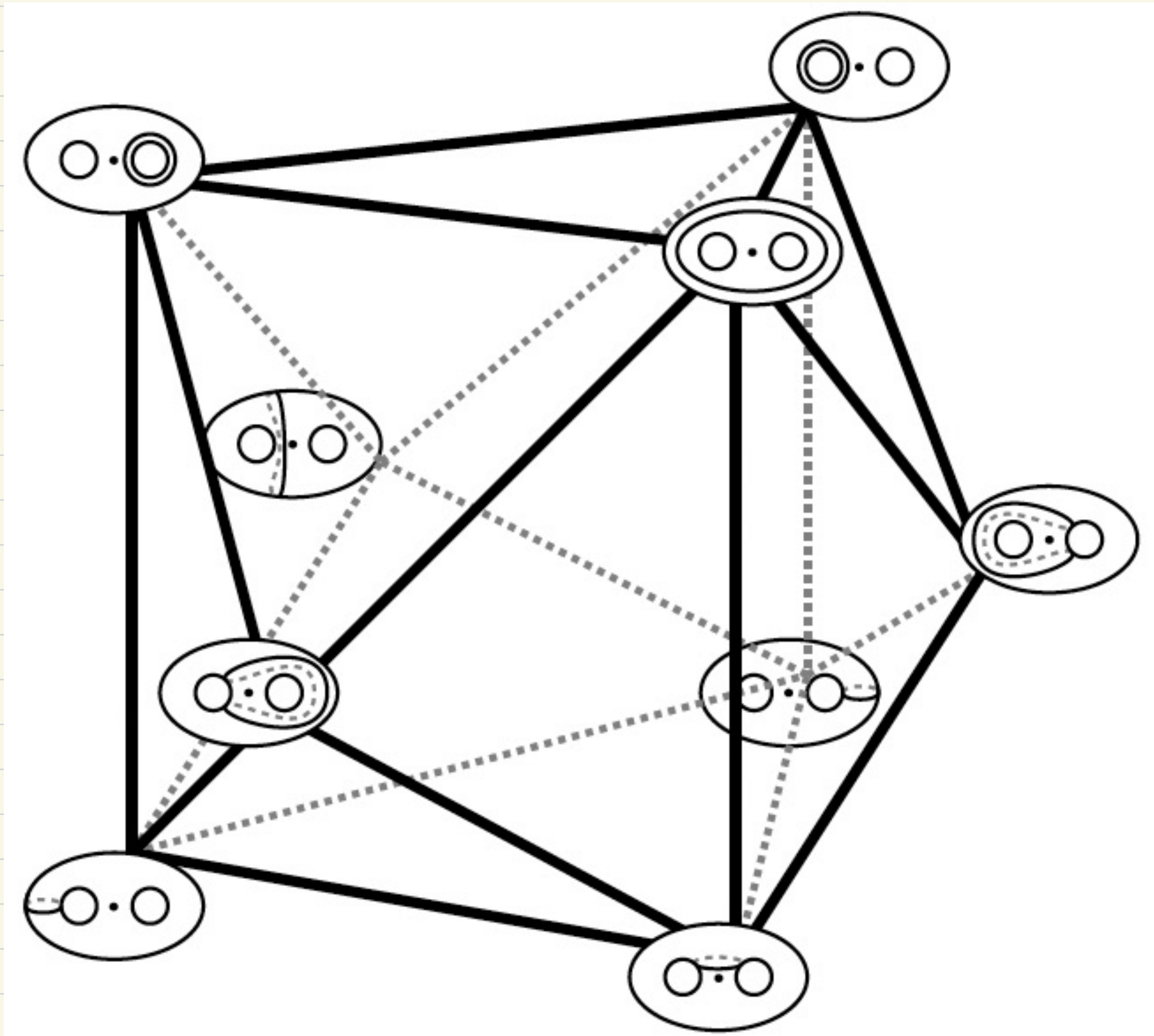
'salient tail of length n'

The unique fully salient diagram generates

Previous relation shows that the original diagram (3) from our vote can be expressed as a linear combination of chord diagrams w/ longer 'salient tails'.

2.19

The Broaddus sphere for $S_{2,1}$:



(courtesy of N. Broaddus)

2.20

Wrapping up

- $S_{g,0}$
- $\underline{g=0}$: Birman–Broaddus-Menasco
- $\underline{(g,n) \; - \; ?}$