Lecture 3:

Cohomology of congruence subgroups

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Goal: Understand for $S = S_{g,n}^{b}$

$$H^*(PMod(S); \mathbb{Q}) \cong H^*(\mathcal{M}(S); \mathbb{Q})$$

Thm (Harer '86, Ivanov '87)

$PMod(S)$ is a virtual duality group of dim

$$v = \begin{cases} 4g-5 & g \geq 2 \quad n=0 \\ 4g+2b+n-4 & g \geq 1 \quad n+b \geq 1 \\ 2b+n-3 & g=0 \end{cases}$$

Stable cohomology:

MMM classes $\kappa_i \in H^{2i}(\mathrm{PMod}(S); \mathbb{Q})$

Euler classes $u_i \in H^2(\mathrm{PMod}(S); \mathbb{Q})$

Madsen-Weiss:

$$\mathbb{Q}[u_1, \ldots, u_n, \kappa_1, \kappa_2, \cdots] \xrightarrow{\cong} H^*(\mathrm{PMod}(S); \mathbb{Q})$$

when $g \gg k$

3.3

Unstable cohomology

Good news: there is lots!

$$\chi(m(S)) >> 0$$

Harer-Zagier S_g

Bini-Harer $S_{g,n}$

Bad news: we don't know many actual classes

Chan-Galatius-Payne '18

Thm (Morita-Sakasai-Suzuki, Church-Farb-Putman)

(*) $H^v(PMod(S); \mathbb{Q}) = 0 \qquad g \geq 2$

Recall: for $\Gamma < PMod(S)$ f.i.

$$\text{vcd}\, PMod(S) := \max\{ k \mid H^k(\Gamma; \underline{A}) \neq 0 \text{ for some } \Gamma\text{-module } A \}$$

Chan Does not hold for $S_{2,n}$ certain n

Guess $g \gg n+b \Rightarrow H^{red}\left[PMod\left(S^{b}_{g,n}\right); \mathbb{Q}\right]$

$= 0$

Next: $n+b \geq 2$ $(g \geq 1)$ (3.14)

$$v = v(g,n,b) = 4g + 2b + n - 4$$

Note: $v' = v(g, n+b, 0) = v(g,1,0) + (n+b-1)$

$$\dim H^v(PMod(S^b_{g,n})[\ell]; \mathbb{Q})$$

$$= \dim H^{v'}(PMod(S^0_{g,n+b})[\ell]; \mathbb{Q})$$

$$\geq \left((n+b-1)\ell^{2g} - 1\right) \dim H^{v(g,1,0) + (n+b-2)}(PMod(S_{g,n+b-1})[\ell]; \mathbb{Q})$$

$$\vdots$$

$$\geq \prod_{k=1}^{n+b-1} (k\ell^{2g} - 1) \dim H^{v(g,1,0)}(PMod(S_{g,1})[\ell]; \mathbb{Q})$$

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Proof of Easier Theorem

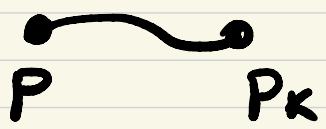
$$H^*(PMod(S_{g,n})[\ell]; \mathbb{Q}) \cong H_*(PMod(S_{g,n})[\ell]; St(S_{g,n}))$$

$$= \left(St(S_{g,n})\right)_{PMod(S_{g,n})[\ell]}$$

$$= \left(\tilde{\mathbb{Q}}[A_p] \otimes St(S_{g,n-1})\right)_{PMod(S_{g,n})[\ell]}$$

$$= \tilde{\mathbb{Q}}[A_p / PMod(S_{g,n})[\ell]] \otimes St(S_{g,n-1})_{PMod[\ell]}$$

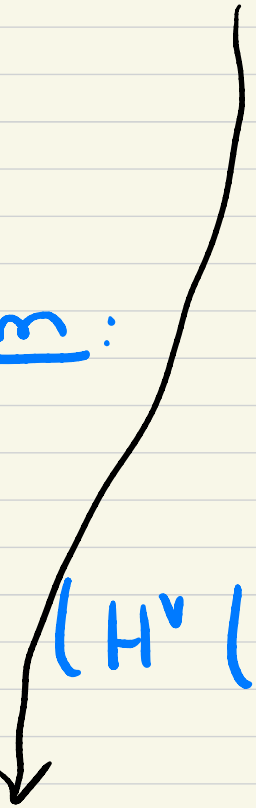
A_p^k



$$A_p = \bigsqcup_{k=1}^{n-1} A_p^k$$

Key fact: $K = \{x \in H_1(S, P);\ \mathbb{Z}/\ell \mid \partial x = p - p_k\}$ (3.16)

has ℓ^{2g} elements.



Easier Theorem: $g \geq 1$, $n \geq 2$ $\quad V = V(g,n,0)$

For all ℓ:

$$\dim \left(H^v(\mathrm{PMod}(S_{g,n})[\ell]; \mathbb{Q}\right)$$

$$\geq \left((n-1)\ell^{2g} - 1\right) \dim H^{v-1}\mathrm{PMod}(S_{g,n-1})[\ell]; \mathbb{Q})$$

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Wrapping up

① Alternate proof: via Birman exact sequence for congruence

② $\mathrm{CohCD}(X) := \max_k \{k \mid H^k(X; F) \neq 0\}$

Get bounds on

$$\mathrm{CohCD}(\mathcal{M}(S_{g,n})) \leq \underbrace{\mathrm{CohCD}(\mathcal{M}(S_g))}_{\geq g-2} + 1$$

Some open questions

Nate's paper has a section 'further questions'

- most relate to S_g vs $S_{g,1}$

- End of § 4.2

$$0 \to \mathcal{I} \to \mathbb{Z}\,\mathrm{Mod}(S_g) \to St(S_g) \to 0$$

Q: Geom int of $\mathcal{I}$?

- $\mathrm{Stab}[\phi_0] < \mathrm{Mod}(S_g)$

(ϕ_0 a generator)