A generating set for the Steinberg module of a Euclidean ring

Talk #2  Review

\( R = \text{ring of integers in number field } F. \)

Virtual Bieri-Eckmann duality: \( \text{Hod}^{-1}(\text{SL}_n; \mathbb{Q}) \cong \text{H}_1(\text{SL}_n; \mathbb{Q} \otimes \mathbb{Z} \text{Stn}(F)) \)

Dualizing module: \( \text{Stn}(F) = \hat{H}^2(n-2)(\text{Tn}(F)) \)

\( \text{Tn}(F) = \text{Tits building} \)

\( \text{vertices} \leftrightarrow \text{proper nonzero subspaces of } F^n \)

\( \text{simplices} \leftrightarrow \text{flags} \)

Thm (Solomon-Tits)

* \( \text{Tn}(F) = \bigvee S^{n-2} \)
* \( \hat{H}^2(n-2)(\text{Tn}(F)) \) is generated by apartment classes.

\( \text{apartments} \leftrightarrow \text{frames } F = l_1 \oplus l_2 \oplus \cdots \oplus l_n \)

\( S(l_1, \ldots, l_n) = \text{full subcomplex of } \text{Tn}(F) \text{ on vertices corresponding to direct sums of lines } l_i. \)

Thm (Ash-Rudolph) \( R \text{ Euclidean} \)

Then \( \hat{H}^2(n-2)(\text{Tn}(F)) \) is gen by integral apartment classes, \( l_i \left[ S(l_1, \ldots, l_n) \right] \) where \( (l_1 \cap R^n) \oplus \cdots \oplus (l_n \cap R^n) = R^n \)

The frame arises from a basis for \( R^n \).

Thm (Lee-Szczyrba) \( R \text{ Euclidean} \)

\( \text{Hod}(\text{SL}_n; \mathbb{Q}) = 0 \)

Last Time: proof that Ash-Rudolph \( \Rightarrow \) Lee-Szczyrba.
Simplified proof of Ash-Rudolph due to Church-Farb-Putman.

Simplicial methods.

**Defn**: A simplicial complex, $\sigma$ in $X \setminus \sigma$ is a simplex, $\sigma = \{s_0, \ldots, s_p\}$.

The link of $\sigma$ is $\text{Link}_X(\sigma) = \text{subcomplex on simplices } \{[t_0, \ldots, t_q] \mid [s_0, \ldots, s_p, t_0, \ldots, t_q] \text{ is a simplex in } X\}$.

**Defn**: An d-dim $\varepsilon$ simplicial complex $X$ is Cohen-Macaulay (CM) if

- $X$ is $(d-1)$-connected
- $\text{Link}_X(\sigma)$ is $(d-2-\dim(\sigma))$-connected, $\forall$ simplices $\sigma$.

**Eq**: the standard simplicial structure on ball, sphere is CM.

**Eq**: Not CM even though contractible.

The original (inequivalent) defn of CM was only a condition on homology of links (not $\dim$).

That decision was a home invariant; this defn due to Quillen is not.

Complex of partial bases

- Fix $R$ - Euclidean ring, $F$ - field of fractions.

**Defn**: $u_0, \ldots, u_p \in R^n$ is a partial basis if it is a subset of a basis (possibly equal to a basis).

**Eq. (Exercise)** $\{u_0\}$ is a partial basis $\iff u_0$ primitive (its components generate $R$).

**Eq. (Exercise)** $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are not partial bases of $\mathbb{Z}^2$.

**Defn**: $Bn(R)$ - complex of partial bases

- Vertices — primitive $u_0 \in R^n$
- Simplices — partial bases.

**Thm (Maazen, 1979)**: Euclidean, the barycentric subdivision of $Bn(R)$ is CM.

$Bn(R)$ is as highly connected as a wedge of $(n-1)$-dim $\varepsilon$ spheres.

**Goal**: Relate Tits building to $Bn(R)$ to get nice generating set for its homology.

**Key**: Quillen's lemma.

**Quillen's lemma**

**Notation**: $X$ poset, $|X|$ geometric realization.

**Lemma (Quillen)**: $f : X \to Y$ strictly increasing map of posets.

- $|Y|$ CM, dim $d$
- $f : |Y| \to |X|$ has $|f| : CM$

Then $|X|$ is CM, and $f^* : \tilde{H}_d(|X|) \to \tilde{H}_d(|Y|)$ surjects.
Proof of Ash-Rudolph

\( y = \text{poset of proper non-zero summands of } F^n \)
\( \text{(So } |Y| = T_n(F), \dim (n-2) \) \)
\( x = \text{proper partial bases of } F^n \text{ under inclusion} \)
\( \text{(So } |X| = \text{barycentric subdivision of } (n-2)\text{-skeleton of } B_n(R) \) \)

\( f: x \rightarrow y \)
\( \langle u_0, \ldots, u_p \rangle \mapsto \text{span}_{F} \langle u_0, \ldots, u_p \rangle \)

Check hypotheses of Lemma:
- map \( f \) is strictly increasing
- \( |Y| \) CH [Solomon = Tits].
- \( \forall U \subseteq F^n, fU = \mathring{U} \) partial bases contained in \( U \) = barycentric subdivision of partial basis complex \( B(U) \).

\( \mathfrak{L} \) Lemma \( \Rightarrow \mathfrak{H}_{n-2}(\mathfrak{L}) \rightarrow \mathfrak{H}_{n-2}(T_n(F)) \) surjects.

\( Eq, n = 3 \)

Since \( B_n(R) \) is \( (n-2) \)-connected, we might expect the deg \((n-2)\) homology of its \((n-2)\)-skeleton to come from the boundaries of the "missing" \((n-1)\) simplices. We will verify this.

Then we're done - the \((n-1)\) simplices correspond to bases of \( R^n \), so their images are integral apartments. By Quillen's lemma, these integral apartment classes generate \( \mathfrak{H}_{n-2}(T_n(F)) \).

From LES of a pair:

\( \cdots \rightarrow \mathfrak{H}_{n-1}(B_n(R), \{x\}) \rightarrow \mathfrak{H}_{n-2}(\{x\}) \rightarrow \mathfrak{H}_{n-2}(B_n(R)) \rightarrow \cdots \)

Surjects by Exactness.

So we have

\( \mathfrak{H}_{n-1}(B_n(R), \{x\}) \rightarrow \mathfrak{H}_{n-2}(\{x\}) \rightarrow \mathfrak{H}_{n-2}(B_n(R)) \)

\( \text{group of } (n-1)\text{-chains on } B_n(R) \)

\( \text{Free abelian gp on bases for } R^n \)

\( \text{since } \{x\} \text { is the codimension-1 skeleton of } B_n(R). \)
Conclusion: $\tilde{H}_{n-2}(T_n(\mathbb{R}))$ is generated by integral apartment classes.

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**Time permitting:** Proof of Maazen, $n=2$.

**Exercise:** It suffices to show $B_2(\mathbb{R})$ is a connected graph.

Let $[\begin{pmatrix} a \\ b \end{pmatrix}] \in B$ be primitive. **Goal:** build path to $[\begin{pmatrix} 0 \\ 1 \end{pmatrix}]$.

**Claim:** We can use the Euclidean algorithm to build an edge $[\begin{pmatrix} a \\ b \end{pmatrix}] - [\begin{pmatrix} d \\ b' \end{pmatrix}]$ with $|d| < |b|$, $1:1$ the Euclidean norm.

**Proof of claim:** Choose any basis $[\begin{pmatrix} a \\ b \end{pmatrix}]$, $[\begin{pmatrix} c' \\ d' \end{pmatrix}]$. $\exists q$ s.t. $|d' - qb| < |b|$

Take $[\begin{pmatrix} c' \\ d' \end{pmatrix}] = [\begin{pmatrix} c' \\ d' \end{pmatrix}] - q[\begin{pmatrix} a \\ b \end{pmatrix}]$. Check: this is still a basis.

Iterate until $d=0$. 