MASTERCLASS EXERCISES, FRIDAY

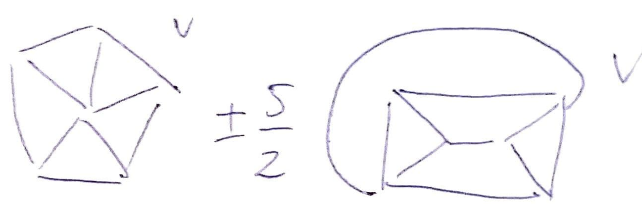
Søren Galatius

Let $C^{(g)}$ be the chain complex from yesterday's lecture, with generators $[\Gamma, w, <]$ with $(\Gamma, w) \in J_g$ and a relation $[\Gamma, w, <] = \pm[\Gamma', w', <']$ for each isomorphism $\phi : (\Gamma, w) \to (\Gamma', w')$ in J_g.

1. Verify that the boundary map is well defined, i.e. that it is compatible with the relations defining the chain complex.

2. Let $W_g \in J_g$ be the "wheel graph" from the lecture (with $w(v) = 0$ for all vertices). Give a description of the group $H = \mathrm{Aut}_{J_g}(W_g)$, and show that it acts on the set $E(W_g)$ by even permutations when $g \geq 3$ is odd, but contains an element acting by an odd permutation when g is even.

3. I drew the following picture today:



It is the sketch of a definition of a cochain $x \in (C^{(5)})^\vee$. It is not quite a complete definition, because the total ordering is not specified. Show that for a suitable choice of orderings, this cochain is a cocycle.

This cocycle then defines a homomorphism $H_*(C^{(5)}) \to \mathbb{Q}$ sending $[W_5] \mapsto 1$, in particular showing that $[W_5] \neq 0$ in homology.

4. Let I be the category of finite sets and injections, let J be a category with finitely many isomorphism classes, and let $X : J^{\mathrm{op}} \to I$ be a functor.

(1) Explain why $J = J_g$ and $X : (\Gamma, w) \mapsto E\Gamma$ is an example.

(2) Let $C(X)$ be the colimit of the composition

$$J \xrightarrow{X} I^{\mathrm{op}} \to \text{Spaces},$$

where the second functor is $T \mapsto \mathbb{R}^T_{\geq 0}$. Convince yourself that this is a locally compact Hausdorff space.

(3) For $n \in \mathbb{N}$, let $J^{\leq n}$ be the full subcategory on those objects $j \in J$ such that $|X(j)| \leq n$, and let $C(X)^{\leq n} \subset C(X)$ be the corresponding subspace. This filtration gives rise to a spectral sequence

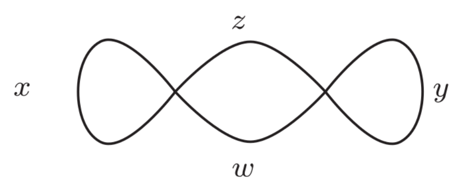
$$E_1^{s,t} = H_c^{s+t}(C(X)^{\leq s}, C(X)^{\leq s-1}; \mathbb{Q}) = H_c^{s+t}(C(X)^{\leq s} \setminus C(X)^{\leq s-1}; \mathbb{Q})$$
$$\Rightarrow H_c^{s+t}(C(X); \mathbb{Q})$$

Explain why this spectral sequence is concentrated on a single horizontal line. Therefore it degenerates at the E_2 page, and gives a cochain complex calculating $H_c^*(C(X); \mathbb{Q})$.

(Hint: this is an abstraction of what I claimed about the example from (a) in the lecture. The basic input is the calculation of $H_c^*(\mathbb{R}^n_{>0}; \mathbb{Q})$ as an representation of S_n, from which one deduces a calculation of $H_c^*(\mathbb{R}^n_{>0}/H; \mathbb{Q})$ for any subgroup $H < S_n$.)

Benjamin Brück

Let G be the graph with two vertices and four edges w, x, y, z depicted below.



In what follows, all subgraphs are edge-induced, i.e. they are determined by specifying subsets of the edge set of G. Recall from the lecture that a *core subgraph* H of G is a proper subgraph such that the fundamental group of each connected component is non-trivial and such that every vertex of H is adjacent to at least two half-edges in H.

(1) Determine the poset of all core subgraphs $C(G)$, where the order relation is given by inclusion. What does the geometric realisation of this poset look like?
 Hint: A good way to draw the geometric realisation of these posets is by considering them as subsets of the barycentric subdivision of a 3-simplex with vertices w, x, y, z.

(2) Describe $X(G)$, the poset of proper, non-empty subgraphs of G where at least one connected component has non-trivial fundamental group. Compare it with $C(G)$.
 (In the lecture, I claimed that $X(G)$ and $C(G)$ are homotopy equivalent. Can you see the retraction $X(G) \to C(G)$?)

(3) What are the subposets of $X(G)$ and $C(G)$ that consist only of *connected* subgraphs?

For a solution, see [1, Figure A.3] (Figure 7 in the arxiv version).

How this relates to the lecture: If we put a labeling on G, then this graph becomes an element of the free splitting complex FS_3 and determines an open 3-simplex in Outer space CV_3. Collapsing a subgraph H of G gives a new element G/H of FS_3 and brings us to a face of the simplex in CV_3. If H has trivial fundamental group, the graph G/H is in the spine L again ("interior of Outer space"), otherwise it is in $FS_3^* = FS_3 \backslash L_3$ ("simplicial boundary"). So the 3-simplex you drew above is a local picture of FS_3. The interior is the open simplex in CV_3 determined by G. The points in the boundary of the simplex that lie in $X(G)$ are in FS_3^* while the remaining points lie in the spine L_3. The translation between these two descriptions is given by

$$\begin{aligned} X(G) &\to FS_3^* \\ H &\mapsto G/H. \end{aligned}$$

References

[1] Benjamin Brück and Radhika Gupta. Homotopy type of the complex of free factors of a free group. *Proceedings of the London Mathematical Society. Third Series*, 121(6):1737–1765, 2020. http://arxiv.org/abs/1810.09380.

Kai-Uwe Bux, Lecture 3

Let U be the 1-skeleton of the 2-dimensional checker board, i.e.:

$$U = \mathbb{R} \times \mathbb{Z} \cup \mathbb{Z} \times \mathbb{R} \subseteq \mathbb{R}^2$$

Note that $\mathbb{Z}^2$ acts on U. Let $M_3 := (U \times U)/\mathbb{Z}^2$ be the orbit space of $U \times U$ mod the diagonal $\mathbb{Z}^2$-action. Describe the homotopy type of M_3 in sufficient detail to show that it has finitely generated homology in dimensions 0 and 1 but infinitely generated homology in dimension 2.