MASTERCLASS EXERCISES

Jeremy, Lecture 1

Let $M_{n,m}$ be the connect sum of n copies of $S^1 \times S^2$ with m open 3-balls removed.

a) Use the Seifert–Van Kampen theorem to show $\pi_1(M_{n,m}) \cong F_n$.

From now on, assume $m > 0$ and fix a basepoint on the boundary. Use this basepoint to define fundamental groups. If X is a connected space and $Y \subset X$ is a subspace, Y is called *non-separating* if $X - Y$ is connected.

b) Suppose $S \subset M_{n,m}$ is an embedded non-separating 2-sphere. Show $\pi_1(M_{n,m} - S) \cong F_{n-1}$. You may assume the group of homeomorphisms of $M_{n,m}$ acts transitively on the set of embedded non-separating 2-spheres.

c) Show $\pi_1(M_{n,m} - S) \to \pi_1(M_{n,m})$ is injective with image a free factor.

Let $Y(M_{n,m})$ be the simplicial complex with p-simplicies given by isotopy classes of disjointly embedded 2-spheres $S_0, \ldots, S_p$ with $S_0 \cup \ldots \cup S_p$ non-separating. Let $Y(F_n)$ be the simplicial complex with vertices free factors $H \subset F_n$ of rank $n-1$ with $[H_1, \ldots, H_{p+1}]$ forming a p-simplex if there is a basis $b_1, \ldots, b_n$ of F_n with each $H_i =< b_1, \ldots, \hat{b_i}, \ldots, b_n >$.

d) Prove $Y(M_{n,1}) \cong Y(F_n)$. You may assume the group of homeomorphisms of $M_{n,m}$ acts transitively on the set of p-simplices of $Y(M_{n,m})$.

e) For m even, identify $Y(M_{n,m})$ as a link in $Y(F_r)$ for some r.

(Extra by the other organizer: Find two elements of F_2 that are a basis of $\mathbb{Z}^2$ after abelianizing.)

Robin Sroka, On the high-dimensional rational cohomology of symplectic groups.

Let $n \geq 1$.

(1) Use Gunnells' theorem to prove that $H^{n^2}(\mathrm{Sp}_{2n}(\mathbb{Z}); \mathbb{Q})$ is zero.
Hint: Jennifer Wilson, Lecture 1.

(2) Work out the details of the proof of Gunnells' theorem for $n = 1$, $\mathrm{Sp}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})$.
Hint: Jennifer Wilson, Lecture 2.

(3) The following is the symplectic analogue of [Jennifer Wilson, Lecture 1, Exercise (12)]. Recall that the symplectic Tits building $\mathcal{T}_n^\omega(\mathbb{Q})$ is the poset of non-trivial isotropic subspaces of the symplectic vector space $(\mathbb{Q}^{2n}, \omega)$. Solomon–Tits theorem states that the symplectic Tits building is homotopy equivalent to a wedge of $(n-1)$-spheres.

$$\mathcal{T}_n^\omega(\mathbb{Q}) \simeq \bigvee S^{n-1}$$

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The goal of this exercise is to explain a discrete Morse theory proof of this theorem, which we learned from [Mladen Bestvina, PL Morse theory, Theorem 5.2.]. If $V \subseteq \mathbb{Q}^{2n}$ is a subspace, we write $V^{\perp} = \{w \in \mathbb{Q}^{2n} : \omega(v, w) = 0 \text{ for all } v \in V\}$ for its symplectic complement.

(a) Verify the theorem for $n = 1$. Assume by induction that the theorem holds for all $m < n$.
(b) Let L be a line in $\mathbb{Q}^{2n}$ and set $X_0 = \{V \in \mathcal{T}_n^{\omega}(\mathbb{Q}) : L \subseteq V\}$.
Prove that X_0 is contractible.
(c) For $1 \leq i \leq n-1$, let $T_i = \{V \in \mathcal{T}_n^{\omega}(\mathbb{Q}) : \dim(V) = i, L \nsubseteq V \text{ and } V \subseteq L^{\perp}\}$ and let X_i be the subposet of $\mathcal{T}_n^{\omega}(\mathbb{Q})$ with underlying set

$$X_i = X_0 \cup T_1 \cup \cdots \cup T_i.$$

Prove that any two distinct elements $V, W \in T_i$ are not comparable and that for any $1 \leq i \leq n-1$:

$$\mathrm{lk}_{\mathcal{T}_n^{\omega}(\mathbb{Q})}(V) \cap X_{i-1} \simeq *.$$

(d) For $1 \leq i \leq n-1$, let $T_{(n-1)+i} = \{V \in \mathcal{T}_n^{\omega}(\mathbb{Q}) : \dim(V) = n-i \text{ and } V \nsubseteq L^{\perp}\}$ and let $X_{(n-1)+i}$ be the subposet of $\mathcal{T}_n^{\omega}(\mathbb{Q})$ with underlying set

$$X_{(n-1)+i} = X_{n-1} \cup T_n \cup \cdots \cup T_{(n-1)+i}.$$

Prove that any two distinct elements $V, W \in T_{(n-1)+i}$ are not comparable for $1 \leq i \leq n-1$ and that for any $1 \leq i \leq n-2$:

$$\mathrm{lk}_{\mathcal{T}_n^{\omega}(\mathbb{Q})}(V) \cap X_{(n-1)+(i-1)} \simeq *.$$

(e) Use the induction hypothesis to prove that for $V \in T_{2n-2}$:

$$\mathrm{lk}_{\mathcal{T}_n^{\omega}(\mathbb{Q})}(V) \cap X_{2n-3} \simeq \bigvee S^{n-2}$$

(f) Use the discrete Morse theory lemma [Jennifer Wilson, Lecture 1, Exercise (11)] to conclude that

$$\mathcal{T}_n^{\omega}(\mathbb{Q}) \simeq \bigvee S^{n-1}.$$

Tara Brendle, Lecture 3

Let $g \geq 0$ and $n \geq 1$ and $\ell \geq 2$, and consider the surface $S_{g,n}$. Recall that $\mathrm{PMod}_{g,n}[\ell]$ denotes the kernel of the action of $\mathrm{PMod}_{g,n}$ on $\mathrm{H}_1(S_{g,n}; \mathbb{Z}/\ell)$. Now let P denote a set of n distinct points on S_g, and identify P with the punctures of $S_{g,n}$. Show that $\mathrm{PMod}_{g,n}[\ell]$ is also the kernel of the action of $\mathrm{PMod}_{g,n}$ on $\mathrm{H}_1(S_g, P; \mathbb{Z}/\ell)$.