MATERCLASS EXERCISES, TUESDAY

Mikala Ørsnes Jansen, Lecture 1

There are two exercises on duality groups (the first is easy) and one (technical) exercise dealing with the construction of the Borel–Serre partial compactification in the special case of $\text{SL}_2$ — this is not difficult but should provide some more understanding of the technical aspects of the construction.

Duality groups.

**Exercise 1.** Suppose $G$ is a duality group of dimension $n$ with respect to a right $G$-module $C$, i.e. there is a class $e \in H^n(G; C)$ such that for any left $G$-module $A$, cap product with $e$ induces an isomorphism $e \cap - : H^i(G; A) \xrightarrow{\sim} H_{n-i}(G; C \otimes A)$ for all $i \in \mathbb{Z}$.

Show that
1. $C \cong H^n(G; \mathbb{Z} G)$ as right $G$-modules, where the cohomology group is a right $G$-module via the right action of $G$ on the coefficients $\mathbb{Z} G$.
2. $H_n(G; C)$ is infinite cyclic generated by $e$.
3. $n = \text{cd } G = \text{hd } G$.

**Exercise 2.** Let $G$ be a group of type FP. Show that the following are equivalent:
1. There is an integer $n$ and a right $G$-module $C$ such that $H^i(G; A) \cong H_{n-i}(G; C \otimes A)$ for all left $G$-modules $A$ and all integers $i$.
2. There is an integer $n$ such that $H^i(G; \mathbb{Z} G \otimes A) = 0$ for all $i \neq n$ and all abelian groups $A$.
3. There is an integer $n$ such that $H^n(G; \mathbb{Z} G) = 0$ for all $i \neq n$ and $H^n(G; \mathbb{Z} G)$ is a free abelian group.
4. There are natural isomorphisms $H^i(G; -) \cong H_{n-i}(G; C \otimes -)$ where $n = \text{cd } G$ and $C = H^n(G; \mathbb{Z} G)$, which are compatible with the connecting homomorphisms in the long exact homology and cohomology sequences associated to a short exact sequence of modules.

**Hints:**
- Recall that induced modules are $H_*$-acyclic.
- For (3) $\Rightarrow$ (4) take a finite projective $G$-resolution $P^* \to \mathbb{Z}$ and consider the dual complex $\text{Hom}_G(P^*, \mathbb{Z} G)$.
- Alternatively/additionally, see Theorem VIII.10.1 in Brown’s *Cohomology of Groups*.

The Borel–Serre partial compactification.

**Exercise 3.** This exercise will exemplify some of the technical aspects of constructing the Borel–Serre partial compactification in the special case of $\text{SL}_2(\mathbb{Z})$.

Consider the space $X = \text{SO}(2) \setminus \text{SL}_2(\mathbb{R})$ with $\text{SL}_2(\mathbb{R})$ acting by right multiplication.

The only (proper) standard parabolic subgroup (i.e. block upper triangular subgroup) of $\text{SL}_2$ is the subgroup $B$ of upper triangular subgroups. The rational parabolic subgroups are the conjugates $\gamma B = \gamma B \gamma^{-1}$ for $\gamma \in \text{SL}_2(\mathbb{Z})$. Let $A = A_B \leq B$ denote the subgroup of diagonal matrices $\left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right)$, $\lambda > 0$.

We have the Iwasawa decomposition: $\text{SL}_2(\mathbb{R}) \cong \text{SO}(2) \times A \times N$, where $N \leq B$ is the subgroup of strict upper triangular matrices and the map from right to left is given by multiplication $(k, a, n) \mapsto kan$.

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The Iwasawa decomposition implies that $B$ acts transitively on $X$ by right multiplication, so that the inclusion $B \hookrightarrow \text{SL}_2(\mathbb{R})$ induces a diffeomorphism

$$(B \cap \text{SO}(2)) \backslash B \overset{\cong}{\to} \text{SO}(2) \backslash \text{SL}_2(\mathbb{R}) = X.$$ 

The geodesic action of $A$ on $X$ is given by left multiplication of $A$ on $(B \cap \text{SO}(2)) \backslash B$ (this is well-defined as $A$ commutes with $B \cap \text{SO}(2)$).

Finally, $X$ is diffeomorphic to the upper half plane $\mathbb{H} = \{ a + bi \mid b > 0 \} \subset \mathbb{C}$ via the map

$$X = \text{SO}(2) \backslash \text{SL}_2(\mathbb{R}) \longrightarrow \mathbb{H}, \quad \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \mapsto \frac{ab + cd + i}{a^2 + c^2}.$$ 

(Note that $a^2 + c^2$ respectively $ab + cd$ is the $(1,1)$- respectively $(1,2)$-entry in the symmetric matrix $g'g.$) This map is equivariant, when $\mathbb{H}$ is equipped with the following action of $\text{SL}_2(\mathbb{Z})$ (you can check this if you like):

$$z. \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \frac{dz + b}{cz + a}, \quad \text{for} \ z \in \mathbb{H}, \ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).$$

In the following exercises you will identify the geodesic action on $\mathbb{H}$, the orbits under this action and how these orbits are partially compactified as a step towards constructing the Borel–Serre partial compactification.

(1) Identify the action of $A$ on $\mathbb{H}$.

(2) Identify the orbits of $A$ and the quotient space $e_B = A \backslash X \cong A \backslash \mathbb{H}$.

(3) Define $\overline{A} := \mathbb{R}_{\geq 0}$ compatibly with the isomorphism $A \overset{\cong}{\to} \mathbb{R}_{>0}$, $(\lambda \ 0 \ \lambda^{-1}) \mapsto \lambda^2$, and identify the partial compactification $\overline{A} \times_A \mathbb{H} = \mathbb{H} \cup e_B$.

We also want to partially compactify all $\text{SL}_2(\mathbb{Z})$-translates of the $A$-orbits (corresponding to the parabolic subgroups $\gamma B$, $\gamma \in \text{SL}_2(\mathbb{Z})$). For a $\gamma \in \text{SL}_n(\mathbb{Z})$, the $\gamma$-translations of the $A$-orbits can also be interpreted as orbits under the following action of $A$ on $X$:

$$\left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) \bullet \gamma x = \left( \begin{array}{cc} \lambda & 0 \\ 0 & \lambda^{-1} \end{array} \right) \bullet (x.\gamma^{-1}) . \gamma,$$

where the $\bullet$ without subscript is the usual geodesic action defined above.

(4) Identify the translates of the $A$-orbits under $\gamma_n = \left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right)$, $n \in \mathbb{Z}$, $\eta = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ and $\mu = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$ (or if you like for a general matrix in $\text{SL}_2(\mathbb{Z})$). Plotting (some of) these orbits in $\mathbb{H}$, you should see why this is called the geodesic action.

(5) How does the partial compactification translate under the $\text{SL}_2(\mathbb{Z})$-action, i.e. what is $\overline{A} \times_A \bullet \mathbb{H}$ for $\gamma \in \text{SL}_2(\mathbb{Z})$? Do this for $\gamma_n$, $\eta$ and $\mu$ above? (What about for general $\gamma \in \text{SL}_2(\mathbb{Z})$?)

Additional remark: We can also identify $X$ with the space $Q$ of positive definite quadratic forms on $\mathbb{R}^2$ via the map

$$X = \text{SO}(2) \backslash \text{SL}_2(\mathbb{R}) \longrightarrow Q, \quad [g] \mapsto (q : x \mapsto x^t (g'g) x).$$

A positive definite quadratic form $q$ can be pictured as an ellipsis of volume 1 in $\mathbb{R}^2$ be plotting the unit ball with respect to the norm induced by $q$. It is interesting to explore the geodesic actions (i.e. also the $\text{SL}_2(\mathbb{Z})$-translated actions) of $A$ on these ellipses. Consider in particular the $A$-orbit of the basepoint corresponding to the standard inner product, that is the standard unit ball (this is $i \in \mathbb{H}$). Under the isomorphism $A \cong \mathbb{R}_{>0}$, tending to zero along this $A$-orbit corresponds to the ellipsis degenerating to a line as the length of the minor axis tends to 0. Different $\text{SL}_2(\mathbb{Z})$-translates give rise to different orientations of the ellipses which in turn result in different lines in $\mathbb{Q}^2$ (corresponding to the major axes). This gives a nice comparison with the Tits building $\mathcal{T}_2.$
Kai-Uwe Bux, Lecture 1

Let $F_n$ be the free group of rank $n$ and let $Y_n$ be the spine of outer space for $F_n$. An automorphism of $F_n$ is determined by what it does on the generators $x_i$.

(1) Show that $\text{Out}(F_n)$ acts on $Y_n$ with compact quotient. More precisely, show that $Y_n$ has exactly one $\text{Out}(F_n)$-orbit of roses and hence is covered by the translates of the star $S_0$ of the vertex given by the standard marked rose.

(2) Call an element $\phi \in \text{Out}(F_n)$ small if $\phi(S_0)$ intersects $S_0$. Let $\Gamma$ be the subgroup of $\text{Out}(F_n)$ generated by small elements. Show that $Y_n$ is connected if and only if $\Gamma = \text{Out}(F_n)$.

(3) Show that the following assignments extend to automorphism of $F_n$:
   • Permutations of the generating set.
   • Sending $x_i$ to its inverse $x_i^{-1}$ and fixing all other $x_j$.
   • Send $x_i$ to $x_ix_j$ (for some $j \neq i$) and fix all the other $x_k$.

(4) Show that the (images of) the automorphisms above generate $\Gamma$.

Tara Brendle, Lecture 1

Let $X$ be any simplicial complex, let $A$ be any discrete set, and let $Y$ be a subset of the join $A \ast X$ with $A, X \subset Y$. Suppose that for all $a \in A$, we have that the inclusion of $\text{lk}_Y(a)$ in $X$ is a homotopy equivalence. Show that the inclusion of $Y$ in $A \ast X$ is also a homotopy equivalence.

HINT: By a result in Hatcher’s book, it suffices to find open covers $\{U_j\}$ of $Y$ and $\{V_j\}$ of $A \ast X$ such that $U_j \subset V_j$ for all $j$, and such that the inclusion of any finite intersection $U_{j_1} \cap \cdots \cap U_{j_k}$ into $V_{j_1} \cap \cdots \cap V_{j_k}$ is always a homotopy equivalence. Use $\{j\}$ and its link in $Y$ to construct $\{U_j\}$.

Extra from the lecture (added by the organizers):
Show that $\mathcal{C}(S_{0,5}) \cong \mathcal{C}(S_{1,2})$, where $\mathcal{C}$ denotes the curve complex of a surface.

Jenny Wilson, Lecture 1

Introduction to the Tits buildings

Let $F$ be a field, and $V$ an $F$-vector space. Recall that the Tits building $T(V)$ is the geometric realization of the poset of proper nonzero vector subspaces of $V$ under inclusion. We sometimes write $T_n(F)$ for $T(F^n)$.

With the following exercises, we can extend the definition of the Tits building to PIDs.

(1) Let $R$ be a PID, and $U \subseteq R^n$ be an $R$-submodule. Show that the following conditions are equivalent. If these conditions hold, we say that $U$ is split or that $U$ is a summand of $R^n$.
   (i) There exist an $R$-submodule $C$ such that $R^n = U \oplus C$.
   (ii) There exists a basis for $U$ that extends to a basis for $R^n$.
   (iii) Any basis for $U$ extends to a basis for $R^n$.
   (iv) The quotient $R^n/U$ is torsion-free.
To define a building, we first need the notion of a Coxeter complex.

Definition (Coxeter complex). Given a Coxeter system \((W,S)\), consider the poset \(P_{W,S}\) of proper standard cosets under reverse inclusion. One way to define the Coxeter complex \(X_{W,S}\) associated to \((W,S)\) is as follows. The \(p\)-simplices of \(X_{W,S}\) are indexed by standard cosets \(wW_J\) with \(|J|=n-p-1\), and assembled in such a way that the geometric realization of \(P_{W,S}\) is the barycentric subdivision of \(X_{W,S}\). In other words, the poset of cells of \(X_{W,S}\) under inclusion is precisely \(P_{W,S}\).
(6) (a) Sketch the Coxeter complex for the symmetric groups $S_2$ and $S_3$.
(b) Describe the standard cosets in the symmetric group $S_{n+1}$.
(c) Show that the Coxeter complex of $S_{n+1}$ can be identified with the flag complex of nonempty subsets of the set $\{1, 2, \ldots, n, n+1\}$.
(d) Show that the Coxeter complex of the symmetric group $S_{n+1}$ can be identified with the boundary of a barycentrically-subdivided $n$-simplex. Conclude that the Coxeter complex is topologically a sphere $S^{n-1}$.

**Definition (Building).** A building is a simplicial complex $\Delta$ that can be written as a union of subcomplexes $\Sigma$, called apartments, that satisfy the following axioms.

(B0) Each apartment $\Sigma$ is a Coxeter complex.

(B1) For any two simplices $A, B \in \Delta$, there is an apartment $\Sigma$ containing both of them.

(B2) If $\Sigma$ and $\Sigma'$ are two apartments containing $A$ and $B$, then there is an isomorphism $\Sigma \cong \Sigma'$ fixing $A$ and $B$ pointwise.

Top-dimensional simplices are called chambers, and codimension-one simplices are panels.

Condition (B2) is equivalent to the following.

(B2') Let $\Sigma$ and $\Sigma'$ be two apartments containing a simplex $C$ that is a chamber of $\Sigma$. Then there is an isomorphism $\Sigma \cong \Sigma'$ fixing every simplex of $\Sigma \cap \Sigma'$.

Let $V$ be a vector space over $\mathbb{Q}$. Recall that the Tits buildings $T(V)$ is the geometric realization of the poset of proper nonzero vector subspaces of $V$.

Further recall that a frame for $V$ is a decomposition $V = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ of $V$ as a direct sum of 1-dimensional subspaces $L_i$. For each frame $L = \{L_1, L_2, \cdots, L_n\}$ for $V$, we define an apartment $A_L$ to be the full subcomplex of $T(V)$ on vertices corresponding to direct sums of all proper nonempty subsets of $\{L_1, L_2, \cdots, L_n\}$.

In the next exercise we will verify that the Tits building $T(V)$, along with the system of apartments $\{A_L \mid L$ a frame for $V\}$, is a building.

(7) (a) Suppose $V$ is $n$-dimensional. Show that an apartment $A_L$ is isomorphic to the Coxeter complex associated to $S_n$. Conclude that axiom (B0) holds.
(b) Given a flag $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_p \subseteq V$, let’s say that frame $L = \{L_1, L_2, \cdots, L_n\}$ is compatible with this flag if every subspace $V_i$ is a direct sum of lines $L_j$. Show that, given any two flags in $V$, there is a frame that is compatible with both of them. Use this result to conclude that (B1) holds.
(Despite being “just” elementary linear algebra, this exercise is not entirely trivial! See Abramenko–Brown “Buildings” Section 4.3.)
(c) Verify that axiom (B2') holds.
  *Hint:* Given a chamber in $\Sigma$ corresponding to a complete flag $0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq V$, construct an explicit isomorphism to the Coxeter complex using the function

$$\phi : \Sigma \longrightarrow \text{subsets of } [n]$$
$$U \longmapsto \{i \mid \dim(U \cap V_i) < \dim(U \cap V_{i+1})\}$$

Observe that this isomorphism depends only on the chamber and not on $\Sigma$. 
Solomon–Tits

The goal of this section is to give a proof of the Solomon–Tits theorem, which states that the Tits building $\mathcal{T}_n(K)$ is homotopy equivalent to a wedge of spheres of dimension $(n - 2)$.

Fix a field $K$ and a positive integer $n$. Recall that the Tits building $\mathcal{T}_n(K)$ is the geometric realization of the poset of proper nonzero subspaces of $K^n$ under inclusion. Explicitly, $\mathcal{T}_n(K)$ is a simplicial complex defined as follows. The vertices of $\mathcal{T}_n(K)$ are proper nonzero subspaces of $K^n$. A collection of vertices span a simplex precisely when they form a flag.

(8) Fix a field $K$. Verify that, when $n = 1$, the building $\mathcal{T}_n(K)$ is empty, and when $n = 2$, the building $\mathcal{T}_n(K)$ is a discrete set of points, that is, a wedge of 0-spheres.

(9) Draw the Tits building for $K = \mathbb{Z}/2\mathbb{Z}$ and $n \leq 3$. Can you see explicitly that it is homotopy equivalent to a wedge of spheres?

To prove the Solomon–Tits theorem, we will use a method sometimes called “discrete Morse theory”. I first learned this proof from Bestvina’s notes [PL Morse Theory].

Definition (Realizations and links). For a poset $T$, write $[T]$ for its geometric realization. For $t \in T$, we write $\text{Lk}_T(t)$ for the link of $t$ in $T$,

$$\text{Lk}_T(t) = \{ s \in T \mid s < t \text{ or } s > t \}.$$

We write $\text{Lk}^1_T(t)$ for the subposet

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(10) Verify that $|\text{Lk}_T(t)| = |\text{Lk}^1_T(t)| \ast |\text{Lk}^1_T(t)|$.

The following result is our key lemma.

Lemma (Discrete Morse Theory). Let $T$ be a poset with $T = X_0 \cup T_1 \cup \cdots \cup T_m$ as sets. Let $X_k = X_0 \cup T_1 \cup \cdots \cup T_k$. Suppose the following:

(i) $|X_0|$ is contractible.

(ii) For $i \geq 1$ then any pair $s, t \in T_i$ of distinct elements are not comparable.

(iii) For $i \geq 1$ and $t \in T_i$,

$$|\text{Lk}_T(t) \cap X_{i-1}| \simeq \bigvee S^{d-1} \quad \text{or} \quad |\text{Lk}_T(t) \cap X_{i-1}| \simeq *.$$

Then $|T|$ is $(d - 1)$-connected. In particular, if $|\text{Lk}_T(t) \cap X_{i-1}| \simeq \bigvee S^{d-1}$ for at least one $i$ and $t$, then $|T|$ is homotopy equivalent to a wedge of $d$-spheres. Otherwise, $|T|$ is contractible.

(11) Prove the Discrete Morse Theory lemma.

(12) Fix a field $K$ and $n \geq 2$. Let $T$ be the poset of nonzero proper subspaces of $K^n$, so $\mathcal{T}_n(K)$ is defined to be $|T|$. Assume by induction that $\mathcal{T}_m(K) \simeq \bigvee S^{m-2}$ for all $m < n$; we proved the base case in Exercise 8. Fix a line $L$ in $K^n$.

- Let $X_0$ be the subposet of $T$ on vertices $V$ such that $L \subseteq V$.
- For $i = 1, \ldots, n - 1$, let $T_i$ be the set of subspaces $V$ of $K^n$

$$T_i = \{ V \subseteq K^n \mid \dim(V) = i, \ L \not\subseteq V \}.$$

(a) Verify that $|X_0| \subseteq |T|$ is the star on the vertex $L$, and hence contractible.

(b) Verify that for fixed $i$, distinct elements in $T_i$ are not comparable.
(c) Suppose \( 1 \leq i \leq (n - 2) \). Show that, for \( V \in T_i \), the subspace \((V + L)\) is a cone point of \([\text{Lk}_T(V) \cap X_{i-1}]\). (Why did we need the assumption \( i \leq (n - 2) \)?)

(d) Verify that, for \( i = n - 1 \) and \( V \in T_i \),

\[ |\text{Lk}_T(V) \cap X_{i-1}| \simeq \bigvee S^{n-3}. \]

**Hint:** Compare \( |\text{Lk}_T(V) \cap X_{i-1}| \) to \( T_{n-1}(K) \).

(e) Use the Discrete Morse Theory lemma to conclude that \( T_n(K) \simeq \bigvee S^{n-2} \).

There are other elegant approaches to computing the homotopy type of \( T_n(K) \). For example, see Abramenko–Brown “Buildings” Section 4.12 for an approach using the theory of shellability. In principle, these proofs can also be used to describe a generating sets for the reduced homology of \( T_n(K) \).

**The Sharbly resolution**

Let \( R \) be a PID, and let \( \text{St}_n(R) \) be the associated Steinberg module,

\[ \text{St}_n(R) := \bar{H}_{n-2}(T_n(R); \mathbb{Z}). \]

By Exercise 5 we can identify \( T_n(R) \) with \( T_n(F) \), where \( F \) is the field of fractions of \( R \).

In this section we will construct a resolution of the Steinberg module due to Lee–Szczarba (1976), which has since been named the **Sharbly resolution**. To quote Lee–Szczarba (Section 4):

In theory, one should be able to use [the Sharbly resolution] to compute the groups \( H_q(\text{SL}_n(R); \text{St}_n(R)) \) for \( q \geq 0 \). However, because of the size of [the terms of the resolution], this is impractical except when \( q = 0 \).

The proof of the Sharbly resolution is both a significant historical development, and also an instructive application of some of the techniques of the field. The construction will use the **Acyclic Covering Lemma** (See Brown “Cohomology of Groups” Section VII Lemma 4.4). Let \( X \) be a CW complex, and suppose \( X \) is the union of a family of nonempty subcomplexes

\[ X = \bigcup_{\alpha \in J} X_{\alpha}. \]

The nerve \( N \) of the family \( X_{\alpha} \) is the abstract simplicial complex with vertex set \( J \) and such that a finite subset \( \sigma \subseteq J \) spans a simplex if and only if the intersection \( X_{\sigma} = \bigcap_{\alpha \in \sigma} X_{\alpha} \) is nonempty.

**Lemma (Acyclic Covering Lemma).** Suppose a CW complex \( X \) is a union of subcomplexes \( X_{\alpha} \) such that every non-empty intersection \( X_{\alpha_0} \cap X_{\alpha_1} \cap \cdots \cap X_{\alpha_p} \) is acyclic. Then \( H_q(X) \cong H_q(N) \), where \( N \) is the nerve of the cover.

(13) **(Bonus).** Use the Mayer–Vietoris spectral sequence to prove the Acyclic Covering Lemma. See Brown “Cohomology of Groups” Section VII.4.

Let \( R \) be a PID. Recall that an element \( v = (r_1, r_2, \ldots, r_n) \in \mathbb{R}^n \) is called **unimodular** if the ideal generated by \( (r_1, r_2, \ldots, r_n) \) is \( R \).

\[ \mathcal{F}_q = \mathcal{F}_q(R^n) = \{(n + q) \times n \text{ matrices } A = (a_{i,j}) \text{ over } R \mid (a_{i,1}, \ldots, a_{i,n}) \text{ is unimodular for all } i\} \]

\[ \mathcal{P}_q = \mathcal{P}_q(R^n) = \{A \in \mathcal{F}_q \mid \text{ each } n \times n \text{ submatrix has determinant } 0\} \]

These sets have actions of \( \text{GL}_n(R) \) by right multiplication. Let \( C(\mathcal{F}_q) \) and \( C(\mathcal{P}_q) \) denote the free abelian groups on \( \mathcal{F}_q \) and \( \mathcal{P}_q \), respectively, and let \( C_q(R^n) = C(\mathcal{F}_q)/C(\mathcal{P}_q) \) be the quotient; it is a right \( \mathbb{Z}[\text{GL}_n(R)]\)-module.
Our goal is to prove the following, Lee–Szczarba Theorem 3.1.

Theorem (The Sharbly resolution). There is an epimorphism \( \phi : C_0(R^n) \to \text{St}_n(R) \) of right \( \mathbb{Z}[\text{GL}_n(R)] \)-modules so that

\[
\ldots \to C_q(R^n) \to C_{q-1}(R^n) \to \cdots \to C_0(R^n) \xrightarrow{\phi} \text{St}_n(R) \to 0
\]

is a free resolution of \( \text{St}_n(R) \) by \( \mathbb{Z}[\text{GL}_n(R)] \)-modules.

\[\text{(14)}\]

(a) Let \( K \) be the simplicial complex whose vertices are the unimodular elements of \( R^n \), and whose simplices are all finite nonempty subsets of vertices. Show that \( K \) is contractible.
(b) Use the long exact sequence of a pair to show that, for a subcomplex \( L \subseteq K \),

\[ H_q(K, L) \cong \tilde{H}_{q-1}(L) \]

(c) Let \( L \subseteq K \) be the subcomplex consisting of all simplices with the property that all of their vertices lie in a proper direct summand of \( R^n \). Let \( \{ H_i \mid i \in I \} \) be the set of direct summands of \( R^n \) of rank \((n-1)\). Let \( K_1 \) be the full subcomplex of \( L \) with vertices lying in \( H_1 \). Show that \( \{ K_i \mid i \in I \} \) is an acyclic covering of \( L \) in the sense of the Acyclic Covering Lemma.

Hint: First argue that, since \( R \) is a PID, a nonempty intersection of summands \( H_i \) must be a direct summand of \( R^n \) isomorphic to \( R^r \) for some \( 0 < r < n \).
(d) Let \( N \) be the nerve of the cover \( \{ K_i \mid i \in I \} \). Use the Acyclic Covering Lemma to deduce that \( H_q(L) \cong H_q(N) \) for all \( q \geq 0 \).
(e) Recall that \( \mathbb{F} \) denotes the field of fractions of \( R \). Let \( \{ W_j \mid j \in J \} \) denote the set of hyperplanes in \( \mathbb{F}^n \). Let \( T_j \) be the subcomplex of the Tits building \( T_n(\mathbb{F}) \) consisting of all simplices with \( W_j \) as a vertex. Show that \( \{ T_j \mid j \in J \} \) is an acyclic covering of \( L \) in the sense of the Acyclic Covering Lemma.
(f) Let \( \tilde{N} \) be the nerve of the cover \( \{ T_j \mid j \in J \} \). Use the Acyclic Covering Lemma to deduce that \( H_q(T_n(\mathbb{F})) \cong H_q(\tilde{N}) \) for all \( q \geq 0 \).
(g) Show that the mapping

\[
\{ \text{summands of } R^n \} \longrightarrow \{ \text{subspaces of } \mathbb{F}^n \cong R^n \otimes_R \mathbb{F} \}
\]

\[
H \longrightarrow H \otimes_R \mathbb{F}
\]

defines a simplicial isomorphism of \( N \) onto \( \tilde{N} \).
(h) We have proved the following isomorphisms. Verify that they are \( \text{GL}_n(R) \)-equivariant.

\[
H_q(K, L) \cong \tilde{H}_{q-1}(L)
\]
\[
\cong \tilde{H}_{q-1}(N)
\]
\[
\cong H_{q-1}(\tilde{N})
\]
\[
\cong \tilde{H}_{q-1}(T_n(\mathbb{F}))
\]

(i) Verify that the \((n-2)\)-skeleton of \( L \) coincides with the \((n-2)\)-skeleton of \( K \), so

\[
C_q(K, L) = 0 \quad \text{for } q \leq n - 2.
\]

(j) Using the Solomon–Tits result that \( T_n(\mathbb{F}) \cong \bigvee S^{n-2} \), deduce that there is an exact sequence

\[
\cdots \longrightarrow C_{q+n}(K, L) \longrightarrow C_{q+n-1}(K, L) \longrightarrow \cdots \longrightarrow C_{-1}(K, L) \longrightarrow \text{St}_n(R) \longrightarrow 0.
\]

(k) Show that there are \( \text{GL}_n(R) \)-equivariant isomorphisms of chain complexes

\[
C_{q+n-1}(K) \cong C(\mathcal{I}_q)
\]
\[
C_{q+n-1}(L) \cong C(\mathcal{I}_q)
\]
\[
C_{q+n-1}(K, L) \cong C_q(R^n)
\]

(l) Conclude the existence of the Sharbly resolution.

(m) Show that differentials in the Sharbly resolution are induced by the map

\[ S_q \rightarrow S_{q-1} \]

\[ A \mapsto \sum (-1)^i d_i(A) \]

where \( d_i \) is the map that deletes the \( i^{th} \) row of the matrix \( A \).

(15) Prove that the Sharbly resolution is free.

Hint: First verify that \( GL_n(R) \) acts freely on the set of \((n + q) \times n\) matrices \( A \) satisfying

- each row of \( A \) is unimodular,
- some \((n \times n)\)-submatrix of \( A \) has non-vanishing determinant.

(16) Give a geometric interpretation of the map \( \phi : C_0(R^n) \rightarrow St_n(R) \). Conclude that the Steinberg module is generated by apartment classes.

Let \( R \) be a Euclidean ring with a multiplicative Euclidean norm. Using the Sharbly resolution, Lee–Szczarba went on to prove that the \( SL_n(R) \)-coinvariants of \( C_0(R^n) \) vanish, which implies that

\[ H_0(SL_n(R); St_n(R)) = 0. \]

By virtual Bieri–Eckmann duality, this then implies that the rational cohomology of \( SL_n(R) \) vanishes in its virtual cohomological dimension.

(17) (Bonus). Let \( R \) be a Euclidean ring with a multiplicative Euclidean norm. Prove directly that \( SL_n(R) \)-coinvariants of \( C_0(R^n) \) vanish. See Lee–Szczarba Theorem 4.1.

**Ash–Rudolph**

Let \( T_n(Q) \) be the Tits building on the rational vector space \( Q^n \). Let \( A_L \) denote the apartment associated to a frame \( L = \{L_1, L_2, \ldots, L_n\} \) for \( Q^n \). Recall the following.

**Definition (Integral apartment).** Let \( L = \{L_1, L_2, \ldots, L_n\} \). The frame \( L \) (and the apartment \( A_L \)) are called integral if

\[ (L_1 \cap Z^n) \oplus (L_2 \cap Z^n) \oplus \cdots \oplus (L_n \cap Z^n) = Z^n. \]

(18) Consider \( Q^2 \). Verify that the apartment corresponding to the frame

\[ \left\{ Q \begin{bmatrix} 1 \\ 0 \end{bmatrix}, Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \]

is integral, but the apartment corresponding to the frame

\[ \left\{ Q \begin{bmatrix} 2 \\ 1 \end{bmatrix}, Q \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \]

is not integral.

(19) Develop a determinant condition for verifying whether or not a frame is integral.

Let \( L = \{L_1, L_2, \ldots, L_n\} \) be a frame, and let \( A_L \) be the associated apartment in the Tits building. Recall from Exercises [6] and [7] that \( A_L \) is an \((n - 2)\)-sphere, specifically, it is simplicially isomorphic to the barycentric subdivision of the boundary of an \((n - 1)\)-simplex.

(20) (a) A permutation \( \sigma \in S_n \) acts on \( L \) by \( L_i \mapsto L_{\sigma(i)} \). Show that \( \sigma \) induces a simplicial isomorphism \( A_L \rightarrow A_{L^\sigma} \). Show that this isomorphism is orientation-preserving if \( \sigma \) is even, and orientation-reversing if \( \sigma \) is odd.
(b) The apartment $A_L$ represents a homology class $[A_L] \in H_{n-2}(T_n(\mathbb{Q}))$. Explain why, to make the sign of $[A_L]$ well-defined, we must order the frame $L$ up to sign. Moreover, the symmetric group $S_n \subseteq \text{GL}_n(\mathbb{Z})$ acts on $[A_L]$ by sign. (Some sources leave $L$ unordered and $[A_L]$ only defined up to sign).

Ash–Ruldolph (1979) proved the following.

**Theorem (Ash–Rudolph).** The homology group $H_{n-2}(T_n(\mathbb{Q}))$ is generated by integral apartment classes

$$\left\{ [A_L] \mid L \text{ an integral frame for } \mathbb{Q}^n, \text{ ordered up to sign} \right\}.$$ 

(21) (a) Find an element of $\text{SL}_n(\mathbb{Z})$ that interchanges the two lines

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

(b) Show that there is no matrix in $\text{SL}_n(\mathbb{Z})$ that interchanges the two lines

$$\left\{ \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$ 

(c) Explain why this non-existence result was an obstacle in our first lecture to computing $H_0(\text{SL}_n(\mathbb{Z}); \text{St}_n(\mathbb{Q}) \otimes \mathbb{Q})$, and explain why it is resolved by Ash–Rudolph.

(22) (Bonus). Again consider $\mathbb{Q}^2$. Write the apartment class corresponding to the frame

$$\left\{ \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$ 

as a linear combination of integral apartment classes.

This theorem of Ash–Rudolph offers a simpler proof of the earlier theorem of Lee–Szczarba.

(23) (a) Let $C_0$ be the free abelian group on the symbols $[L]$ with $L = \{L_0, L_1, \ldots, L_n\}$ a frame for $\mathbb{R}^n$, $L$ ordered up to sign, subject to the relation

$$\sigma \cdot [L] = (-1)^{\text{sign}(\sigma)}[L] \quad \text{for } \sigma \in S_n.$$ 

Show that the $\text{SL}_n(\mathbb{Z})$-coinvariants of $C_0$ vanish.

(b) Deduce that $H_0(\text{SL}_n(\mathbb{Z}); \text{St}_n(\mathbb{Q})) = 0$.

(c) Deduce that $H^{\text{ecd}}(\text{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0$. 