MASTERCLASS EXERCISES, TUESDAY

Mikala Ørsnes Jansen, Lecture 1

There are two exercises on duality groups (the first is easy) and one (technical) exercise dealing with the construction of the Borel–Serre partial compactification in the special case of SL_2 — this is not difficult but should provide some more understanding of the technical aspects of the construction.

Duality groups.

Exercise 1. Suppose G is a duality group of dimension n with respect to a right G-module C, i.e. there is a class $e \in H_n(G; C)$ such that for any left G-module A, cap product with e induces an isomorphism

$$e \cap - : H^i(G; A) \xrightarrow{\cong} H_{n-i}(G; C \otimes A) \qquad \text{for all } i \in \mathbb{Z}.$$

Show that

1. $C \cong H^n(G; \mathbb{Z}G)$ as right G-modules, where the cohomology group is a right G-module via the right action of G on the coefficients $\mathbb{Z}G$.
2. $H_n(G; C)$ is infinite cyclic generated by e.
3. $n = \mathrm{cd}\, G = \mathrm{hd}\, G$.

Exercise 2. Let G be a group of type FP. Show that the following are equivalent:

1. There is an integer n and a right G-module C such that $H^i(G; A) \cong H_{n-i}(G; C \otimes A)$ for all left G-modules A and all integers i.
2. There is an integer n such that $H^i(G; \mathbb{Z}G \otimes A) = 0$ for all $i \neq n$ and all abelian groups A.
3. There is an integer n such that $H^i(G; \mathbb{Z}G) = 0$ for all $i \neq n$ and $H^n(G; \mathbb{Z}G)$ is a free abelian group.
4. There are natural isomorphisms $H^i(G; -) \cong H_{n-i}(G; C \otimes -)$ where $n = \mathrm{cd}\, G$ and $C = H^n(G; \mathbb{Z}G)$, which are compatible with the connecting homomorphisms in the long exact homology and cohomology sequences associated to a short exact sequence of modules.

[*Hints:*

- Recall that induced modules are H_*-acyclic.
- For (3) $\Rightarrow$ (4) take a finite projective G-resolution $P^* \to \mathbb{Z}$ and consider the dual complex $\mathrm{Hom}_G(P^*, \mathbb{Z}G)$.
- Alternatively/additionally, see Theorem VIII.10.1 in Brown's *Cohomology of Groups*.]

The Borel–Serre partial compactification.

Exercise 3. This exercise will exemplify some of the technical aspects of constructing the Borel–Serre partial compactification in the special case of $\mathrm{SL}_2(\mathbb{Z})$.

Consider the space $X = \mathrm{SO}(2) \backslash \mathrm{SL}_2(\mathbb{R})$ with $\mathrm{SL}_2(\mathbb{R})$ acting by right multiplication.

The only (proper) standard parabolic subgroup (i.e. block upper triangular subgroup) of SL_2 is the subgroup B of upper triangular subgroups. The rational parabolic subgroups are the conjugates ${}^{\gamma}B = \gamma B \gamma^{-1}$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. Let $A = A_B \leq B$ denote the subgroup of diagonal matrices $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, $\lambda > 0$.

We have the Iwasawa decomposition: $\mathrm{SL}_2(\mathbb{R}) \cong \mathrm{SO}(2) \times A \times N$, where $N \leq B$ is the subgroup of strict upper triangular matrices and the map from right to left is given by multiplication $(k, a, n) \mapsto kan$.

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The Iwasawa decomposition implies that B acts transitively on X by right multiplication, so that the inclusion $B \hookrightarrow \mathrm{SL}_2(\mathbb{R})$ induces a diffeomorphism

$$(B \cap \mathrm{SO}(2))\backslash B \xrightarrow{\cong} \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R}) = X.$$

The *geodesic action* of A on X is given by left multiplication of A on $(B \cap \mathrm{SO}(2))\backslash B$ (this is well-defined as A commutes with $B \cap \mathrm{SO}(2)$).

Finally, X is diffeomorphic to the upper half plane $\mathbb{H} = \{a + bi \mid b > 0\} \subset \mathbb{C}$ via the map

$$X = \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathbb{H}, \qquad \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] \mapsto \frac{ab + cd + i}{a^2 + c^2}.$$

(Note that $a^2 + c^2$ respectively $ab + cd$ is the $(1,1)$- respectively $(1,2)$-entry in the symmetric matrix $g^t g$.) This map is equivariant, when $\mathbb{H}$ is equipped with the following action of $\mathrm{SL}_2(\mathbb{Z})$ (you can check this if you like):

$$z.\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \frac{dz + b}{cz + a}, \qquad \text{for } z \in \mathbb{H},\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

In the following exercises you will identify the geodesic action on $\mathbb{H}$, the orbits under this action and how these orbits are partially compactified as a step towards constructing the Borel–Serre partial compactification.

(1) Identify the action of A on $\mathbb{H}$.
(2) Identify the orbits of A and the quotient space $e_B = A\backslash X \cong A\backslash \mathbb{H}$.
(3) Define $\overline{A} := \mathbb{R}_{\geq 0}$ compatibly with the isomorphism $A \xrightarrow{\cong} \mathbb{R}_{>0}$, $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \mapsto \lambda^2$, and identify the partial compactification $\overline{A} \times_A \mathbb{H} = \mathbb{H} \cup e_B$.

We also want to partially compactify all $\mathrm{SL}_2(\mathbb{Z})$-translates of the A-orbits (corresponding to the parabolic subgroups ${}^{\gamma}B$, $\gamma \in \mathrm{SL}_2(\mathbb{Z})$). For a $\gamma \in \mathrm{SL}_n(\mathbb{Z})$, the γ-translations of the A-orbits can also be interpreted as orbits under the following action of A on X:

$$\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \bullet_\gamma x = \Big(\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \bullet (x.\gamma^{-1})\Big).\gamma,$$

where the $\bullet$ without subscript is the usual geodesic action defined above.

(4) Identify the translates of the A-orbits under $\gamma_n = \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$, $n \in \mathbb{Z}$, $\eta = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mu = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ (or if you like for a general matrix in $\mathrm{SL}_2(\mathbb{Z})$). Plotting (some of) these orbits in $\mathbb{H}$, you should see why this is called the geodesic action.
(5) How does the partial compactification translate under the $\mathrm{SL}_2(\mathbb{Z})$-action, i.e. what is $\overline{A} \times_{A,\bullet_\gamma} \mathbb{H}$ for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$? Do this for γ_n, η and μ above? (What about for general $\gamma \in \mathrm{SL}_2(\mathbb{Z})$?)

Additional remark: We can also identify X with the space $\mathcal{Q}$ of positive definite quadratic forms on $\mathbb{R}^2$ via the map

$$X = \mathrm{SO}(2)\backslash \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathcal{Q}, \qquad [g] \mapsto (q\colon x \mapsto x^t(g^t g)x)$$

A positive definite quadratic form q can be pictured as an ellipsis of volume 1 in $\mathbb{R}^2$ be plotting the unit ball with respect to the norm induced by q. It is interesting to explore the geodesic actions (i.e. also the $\mathrm{SL}_2(\mathbb{Z})$-translated actions) of A on these ellipses. Consider in particular the A-orbit of the basepoint corresponding to the standard inner product, that is the standard unit ball (this is $i \in \mathbb{H}$). Under the isomorphism $A \cong \mathbb{R}_{>0}$, tending to zero along this A-orbit corresponds to the ellipsis degenerating to a line as the length of the minor axis tends to 0. Different $\mathrm{SL}_2(\mathbb{Z})$-translates give rise to different orientations of the ellipses which in turn result in different lines in $\mathbb{Q}^2$ (corresponding to the major axes). This gives a nice comparison with the Tits building $\mathcal{T}_2$.

Kai-Uwe Bux, Lecture 1

Let F_n be the free group of rank n and let Y_n be the spine of outer space for F_n. Fix a free generating set $\{x_1, \ldots, x_n\}$ of F_n. An automorphism of F_n is determined by what it does on the generators x_i.

(1) Show that $\mathrm{Out}(F_n)$ acts on Y_n with compact quotient. More precisely, show that Y_n has exactly one $\mathrm{Out}(F_n)$-orbit of roses and hence is covered by the translates of the star S_0 of the vertex given by the standard marked rose.
(2) Call an element $\phi \in \mathrm{Out}(F_n)$ *small* if $\phi(S_0)$ intersects S_0. Let Γ be the subgroup of $\mathrm{Out}(F_n)$ generated by *small elements*. Show that Y_n is connected if and only if $\Gamma = \mathrm{Out}(F_n)$.
(3) Show that the following assignments extend to automorphism of F_n:
  - Permutations of the generating set.
  - Sending x_i to its inverse x_i^{-1} and fixing all other x_j.
  - Send x_i to $x_i x_j$ (for some $j \neq i$) and fix all the other x_k.

(4) Show that the (images of) the automorphisms above generate Γ.

Tara Brendle, Lecture 1

Let X be any simplicial complex, let A be any discrete set, and let Y be a subset of the join $A * X$ with $A, X \subset Y$. Suppose that for all $a \in A$, we have that the inclusion of $\mathrm{lk}_Y(a)$ in X is a homotopy equivalence. Show that the inclusion of Y in $A * X$ is also a homotopy equivalence.

HINT: By a result in Hatcher's book, it suffices to find open covers $\{U_j\}$ of Y and $\{V_j\}$ of $A * X$ such that $U_j \subset V_j$ for all j, and such that the inclusion of any finite intersection $U_{j_1} \cap \cdots, U_{j_k}$ into $V_{j_1} \cap \cdots, V_{j_k}$ is always a homotopy equivalence. Use $\{j\}$ and its link in Y to construct $\{U_j\}$.

Extra from the lecture (added by the organizers):
Show that $\mathcal{C}(S_{0,5}) \cong \mathcal{C}(S_{1,2})$, where $\mathcal{C}$ denotes the curve complex of a surface.

Jenny Wilson, Lecture 1

Introduction to the Tits buildings

Let $\mathbb{F}$ be a field, and V an $\mathbb{F}$-vector space. Recall that the *Tits building* $\mathcal{T}(V)$ is the geometric realization of the poset of proper nonzero vector subspaces of V under inclusion. We sometimes write $\mathcal{T}_n(\mathbb{F})$ for $\mathcal{T}(\mathbb{F}^n)$.

With the following exercises, we can extend the definition of the Tits building to PIDs.

(1) Let R be a PID, and $U \subseteq R^n$ be an R-submodule. Show that the following conditions are equivalent. If these conditions hold, we say that U is *split* or that U is a *summand* of R^n.
  (i) There exist an R-submodule C such that $R^n = U \oplus C$.
  (ii) There exists a basis for U that extends to a basis for R^n.
  (iii) Any basis for U extends to a basis for R^n.
  (iv) The quotient R^n/U is torsion-free.

(2) Let R be a PID. An element $v = (r_1, r_2, \ldots, r_n) \in R^n$ is called *unimodular* if the ideal generated by $(r_1, r_2, \ldots, r_n)$ is R. In other words, the gcd of $(r_1, r_2, \ldots, r_n)$ is a unit.
  (a) Show that a nonzero element $v \in R^n$ spans a direct summand of R^n if and only if it is unimodular.
  (b) Show that $v \in R^n$ is an element of a basis for R^n if and only if it is unimodular.
  (c) Give an example of a PID R and two unimodular vectors in R^n that can never both be elements of the same basis.

For R a PID and V a free R-module, we may further define $\mathcal{T}(V)$ to be the poset of prroper nonzero summands of V under inclusion.

(3) Let R be a PID, and let U, V be summands of R^n.
  (a) Show that $U \cap V$ is always a summand of R^n.
  (b) Show that $U + V$ need not be a summand of R^n.

(4) Let $U \subseteq R^n$ be a direct summand, and let $W \subseteq U$. Show that W is a summand of U if and only if it is a summand of R^n.

In the next exercises we will that this ostensible generalizations of the Tits buildings to PIDs in fact reduces to the field case.

(5) Let R be a PID and $F(R)$ be its field of fractions.
  (a) Show that there is a bijection between R-submodule summands of R^n and $F(R)$-vector subspaces of $F(R)^n$, given by the following correspondence. Identify $F(R)^n$ with $R^n \otimes_R F(R)$.
$$\{\text{summands of } R^n\} \longleftrightarrow \{\text{subspaces of } F(R)^n\}$$
$$U \longmapsto U \otimes_R F(R)$$
$$V \cap R^n \longleftarrow V$$
  (b) Verify that this bijection induces an isomorphism of posets of submodules under inclusion.
  (c) Conclude that $\mathcal{T}_n(R)$ can be canonically identified with $\mathcal{T}_n(F(R))$.

Coxeter complexes and buildings

The Tits buildings, as the name suggests, are examples of *buildings*. In this exercise we will define a building and verify that the Tits buildings satisfy the axioms.

To define a building, we first need the notion of a Coxeter complex.

Definition (Coxeter system). A *Coxeter system* is a group W (the *Coxeter group*) along with a distinguished generating set $S = \{s_1, s_2, \ldots, s_n\} \subseteq W$ such that the corresponding presentation has the form
$$W = \langle s_1, s_2, \ldots, s_n \mid (s_i s_j)^{m_{i,j}} = 1 \rangle, \qquad m_{i,i} = 1, \quad 2 \leq m_{i,j} \leq \infty \text{ for } i \neq j.$$
Coxeter groups are abstract generalizations of *reflection groups*. Notably for our purposes, the symmetric group S_{n+1} is a Coxeter group with generators the simple transpositions $s_i = (i\; i+1)$ and associated presentation
$$S_{n+1} = \langle s_1, s_2, \ldots, s_n \mid s_i^2,\ (s_i s_{i+1})^3,\ (s_i s_j)^2 \text{ for } |i-j| > 1 \rangle.$$
Given a Coxeter system (W, S), a *standard subgroup* of W is any subgroup W_J generated by a subset J of S. A *standard coset* is a coset wW_J for $w \in W$ and W_J a standard coset.

Definition (Coxeter complex). Given a Coxeter system (W, S), consider the poset $P_{W,S}$ of proper standard cosets under reverse inclusion. One way to define the *Coxeter complex* $X_{W,S}$ associated to (W, S) is as follows. The p-simplices of $X_{W,S}$ are indexed by standard cosets wW_J with $|J| = n - p - 1$, and assembled in such a way that the geometric realization of $P_{W,S}$ is the barycentric subdivision of $X_{W,S}$. In other words, the poset of cells of $X_{W,S}$ under inclusion is precisely $P_{W,S}$.

(6) (a) Sketch the Coxeter complex for the symmetric groups S_2 and S_3.
(b) Describe the standard cosets in the symmetric group S_{n+1}.
(c) Show that the Coxeter complex of S_{n+1} can be identified with the flag complex of nonempty subsets of the set $\{1, 2, \ldots, n, n+1\}$.
(d) Show that the Coxeter complex of the symmetric group S_{n+1} can be identified with the boundary of a barycentrically-subdivided n-simplex. Conclude that the Coxeter complex is topologically a sphere S^{n-1}.

Definition (Building). A *building* is a simplicial complex Δ that can be written as a union of subcomplexes Σ, called *apartments*, that satisfy the following axioms.
(B0) Each apartment Σ is a Coxeter complex.
(B1) For any two simplices A, $B \in \Delta$, there is an apartment Σ containing both of them.
(B2) If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

Top-dimensional simplices are called *chambers*, and codimension-one simplices are *panels*.

Condition **(B2)** is equivalent to the following.

(B2') Let Σ and Σ' be two apartments containing a simplex C that is a chamber of Σ. Then there is an isomorphism $\Sigma \xrightarrow{\cong} \Sigma'$ fixing every simplex of $\Sigma \cap \Sigma'$.

Let V be a vector space over $\mathbb{Q}$. Recall that the *Tits buildings* $\mathcal{T}(V)$ is the geometric realization of the poset of proper nonzero vector subspaces of V.

Further recall that a *frame* for V is a decomposition $V = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ of V as a direct sum of 1-dimensional subspaces L_i. For each frame $L = \{L_1, L_2, \cdots, L_n\}$ for V, we define an apartment A_L to be the full subcomplex of $\mathcal{T}(V)$ on vertices corresponding to direct sums of all proper nonempty subsets of $\{L_1, L_2, \cdots, L_n\}$.

In the next exercise we will verify that the Tits building $\mathcal{T}(V)$, along with the system of apartments $\{A_L \mid L \text{ a frame for } V\}$, is a building.

(7) (a) Suppose V is n-dimensional. Show that an apartment A_L is isomorphic to the Coxeter complex associated to S_n. Conclude that axiom **(B0)** holds.
(b) Given a flag $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_p \subsetneq V$, let's say that frame $L = \{L_1, L_2, \cdots, L_n\}$ is *compatible* with this flag if every subspace V_i is a direct sum of lines L_j. Show that, given any two flags in V, there is a frame that is compatible with both of them. Use this result to conclude that **(B1)** holds.
(Despite being "just" elementary linear algebra, this exercise is not entirely trivial! See Abramenko–Brown "Buildings" Section 4.3.)
(c) Verify that axiom **(B2')** holds.
Hint: Given a chamber in Σ corresponding to a complete flag $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V$, construct an explicit isomorphism to the Coxeter complex using the function

$$\begin{aligned} \phi : \Sigma &\longrightarrow \{\text{subsets of } [n]\} \\ U &\longmapsto \{i \mid \dim(U \cap V_i) < \dim(U \cap V_{i+1})\} \end{aligned}$$

Observe that this isomorphism depends only on the chamber and not on Σ.

Solomon–Tits

The goal of this section is to give a proof of the Solomon–Tits theorem, which states that the Tits building $\mathcal{T}_n(K)$ is homotopy equivalent to a wedge of spheres of dimension $(n-2)$.

Fix a field K and a positive integer n. Recall that the Tits building $\mathcal{T}_n(K)$ is the geometric realization of the poset of proper nonzero subspaces of K^n under inclusion. Explicitly, $\mathcal{T}_n(K)$ is a simplicial complex defined as follows. The vertices of $\mathcal{T}_n(K)$ are proper nonzero subspaces of K^n. A collection of vertices span a simplex precisely when they form a flag.

(8) Fix a field K. Verify that, when $n=1$, the building $\mathcal{T}_n(K)$ is empty, and when $n=2$, the building $\mathcal{T}_n(K)$ is a discrete set of points, that is, a wedge of 0-spheres.
(9) Draw the Tits building for $K=\mathbb{Z}/2\mathbb{Z}$ and $n\leq 3$. Can you see explicitly that it is homotopy equivalent to a wedge of spheres?

To pove the Solomon–Tits theorem, will use a method sometimes called "discrete Morse theory". I first learned this proof from Bestvina's notes "PL Morse Theory".

Definition (Realizations and links). For a poset T, write $|T|$ for its geometric realization. For $t\in T$, we write $\mathrm{Lk}_T(t)$ for the link of t in T,

$$\mathrm{Lk}_T(t)=\{s\in T \mid s<t \text{ or } s>t\}.$$

We write $\mathrm{Lk}^{\uparrow}_T(t)$ for the subposet

$$\mathrm{Lk}^{\uparrow}_T(t)=\{s\in T \mid s>t\}$$

and we write $\mathrm{Lk}^{\downarrow}_T(t)$ for the subposet

$$\mathrm{Lk}^{\downarrow}_T(t)=\{s\in T \mid s<t\}.$$

(10) Verify that $|\mathrm{Lk}_T(t)|=|\mathrm{Lk}^{\uparrow}_T(t)| \underset{\text{join}}{*} |\mathrm{Lk}^{\downarrow}_T(t)|$.

The following result is our key lemma.

Lemma (Discrete Morse Theory). Let T be a poset with $T=X_0\cup T_1\cup\cdots\cup T_m$ as sets. Let $X_k=X_0\cup T_1\cup\cdots\cup T_k$. Suppose the following:
(i) $|X_0|$ is contractible.
(ii) For $i\geq 1$ then any pair $s,t\in T_i$ of distinct elements are not comparable.
(iii) For $i\geq 1$ and $t\in T_i$,

$$|\mathrm{Lk}_T(t)\cap X_{i-1}|\simeq \bigvee S^{d-1} \qquad \text{or} \qquad |\mathrm{Lk}_T(t)\cap X_{i-1}|\simeq *.$$

Then $|T|$ is $(d-1)$-connected. In particular, if $|\mathrm{Lk}_T(t)\cap X_{i-1}|\simeq\bigvee S^{d-1}$ for at least one i and t, then $|T|$ is homotopy equivalent to a wedge of d-spheres. Otherwise, $|T|$ is contractible.

(11) Prove the Discrete Morse Theory lemma.
(12) Fix a field K and $n\geq 2$. Let T be the poset of nonzero proper subspaces of K^n, so $\mathcal{T}_n(K)$ is defined to be $|T|$. Assume by induction that $\mathcal{T}_m(K)\simeq\bigvee S^{m-2}$ for all $m<n$; we proved the base case in Exercise (8). Fix a line L in K^n.
   - Let X_0 be the subposet of T on vertices V such that $L\subseteq V$.
   - For $i=1,\ldots,n-1$, let T_i be the set of subspaces V of K^n

   $$T_i=\{V\subseteq K^n \mid \dim(V)=i,\ L\not\subseteq V\ \}$$

   (a) Verify that $|X_0|\subseteq|T|$ is the star on the vertex L, and hence contractible.
   (b) Verify that for fixed i, distinct elements in T_i are not comparable.

(c) Suppose $1 \leq i \leq (n-2)$. Show that, for $V \in T_i$, the subspace $(V + L)$ is a cone point of $|\mathrm{Lk}_T(V) \cap X_{i-1}|$. (Why did we need the assumption $i \leq (n-2)$?)
(d) Verify that, for $i = n-1$ and $V \in T_i$,

$$|\mathrm{Lk}_T(V) \cap X_{i-1}| \simeq \bigvee S^{n-3}.$$

Hint: Compare $|\mathrm{Lk}_T(V) \cap X_{i-1}|$ to $\mathcal{T}_{n-1}(K)$.
(e) Use the Discrete Morse Theory lemma to conclude that $\mathcal{T}_n(K) \simeq \bigvee S^{n-2}$.

There are other elegant approaches to computing the homotopy type of $\mathcal{T}_n(K)$. For example, see Abramenko–Brown "Buildings" Section 4.12 for an approach using the theory of shellability. In principle, these proofs can also be used to describe a generating sets for the reduced homology of $\mathcal{T}_n(K)$.

The Sharbly resolution

Let R be a PID, and let $\mathrm{St}_n(R)$ be the associated Steinberg module,

$$\mathrm{St}_n(R) := \widetilde{H}_{n-2}(\mathcal{T}_n(R); \mathbb{Z}).$$

By Exercise 5, we can identify $\mathcal{T}_n(R)$ with $\mathcal{T}_n(\mathbb{F})$, where $\mathbb{F}$ is the field of fractions of R.

In this section we will construct a resolution of the Steinberg module due to Lee–Szczarba (1976), which has since been named the *Sharbly resolution.* To quote Lee–Szczarba (Section 4):

> *In theory, one should be able to use [the Sharbly resolution] to compute the groups $H_q(\mathrm{SL}_n(R); \mathrm{St}_n(R))$ for $q \geq 0$. However, because of the size of [the terms of the resolution], this is impractical except when $q = 0$.*

The proof of the Sharbly resolution is both a significant historical development, and also an instructive application of some of the techniques of the field. The construction will use the *Acyclic Covering Lemma* (See Brown "Cohomology of Groups" Section VII Lemma 4.4). Let X be a CW complex, and suppose X is the union of a family of nonempty subcomplexes

$$X = \bigcup_{\alpha \in J} X_\alpha.$$

The *nerve* N of the family X_α is the abstract simplicial complex with vertex set J and such that a finite subset $\sigma \subseteq J$ spans a simplex if and only if the intersection $X_\sigma = \bigcap_{\alpha \in \sigma} X_\alpha$ is nonempty.

Lemma (Acyclic Covering Lemma). Suppose a CW complex X is a union of subcomplexes X_α such that every non-empty intersection $X_{\alpha_0} \cap X_{\alpha_1} \cap \cdots \cap X_{\alpha_p}$ is acyclic. Then $H_*(X) \cong H_*(N)$, where N is the nerve of the cover.

(13) **(Bonus).** Use the Mayer–Vietoris spectral sequence to prove the Acyclic Covering Lemma. See Brown "Cohomology of Groups" Section VII.4.

Let R be a PID. Recall that an element $v = (r_1, r_2, \ldots, r_n) \in R^n$ is called *unimodular* if the ideal generated by $(r_1, r_2, \ldots, r_n)$ is R.

$$\mathscr{S}_q = \mathscr{S}_q(R^n) = \{(n+q) \times n \text{ matrices } A = (a_{i,j}) \text{ over } R \mid (a_{i,1}, \ldots, a_{i,n}) \text{ is unimodular for all } i\}$$
$$\mathscr{P}_q = \mathscr{P}_q(R^n) = \{A \in \mathscr{S}_q \mid \text{ each } n \times n \text{ submatrix has determinant } 0\}$$

These sets have actions of $\mathrm{GL}_n(R)$ by right multiplication. Let $C(\mathscr{S}_q)$ and $C(\mathscr{P}_q)$ denote the free abelian groups on $\mathscr{S}_q$ and $\mathscr{P}_q$, respectively, and let $C_q(R^n) = C(\mathscr{S}_q)/C(\mathscr{P}_q)$ be the quotient; it is a right $\mathbb{Z}[\mathrm{GL}_n(R)]$-module.

Our goal is to prove the following, Lee–Szczarba Theorem 3.1.

Theorem (The Sharbly resolution). There is an epimorphism $\phi : C_0(R^n) \to \mathrm{St}_n(R)$ of right $\mathbb{Z}[\mathrm{GL}_n(R)]$-modules so that

$$\longrightarrow C_q(R^n) \longrightarrow C_{q-1}(R^n) \longrightarrow \cdots \longrightarrow C_0(R^n) \xrightarrow{\phi} \mathrm{St}_n(R) \longrightarrow 0$$

is a free resolution of $\mathrm{St}_n(R)$ by $\mathbb{Z}[\mathrm{GL}_n(R)]$-modules.

(14) (a) Let K be the simplicial complex whose vertices are the unimodular elements of R^n, and whose simplices are all finite nonempty subsets of vertices. Show that K is contractible.

(b) Use the long exact sequence of a pair to show that, for a subcomplex $L \subseteq K$,

$$H_q(K, L) \cong \widetilde{H}_{q-1}(L).$$

(c) Let $L \subseteq K$ be the subcomplex consisting of all simplices with the property that all of their vertices lie in a proper direct summand of R^n. Let $\{H_i \mid i \in I\}$ be the set of direct summands of R^n of rank $(n-1)$. Let K_i be the full subcomplex of L with vertices lying in H_i. Show that $\{K_i \mid i \in I\}$ is an acyclic covering of L in the sense of the Acyclic Covering Lemma.

Hint: First argue that, since R is a PID, a nonempty intersection of summands H_i must be a direct summand of R^n isomorphic to R^r for some $0 < r < n$.

(d) Let N be the nerve of the cover $\{K_i \mid i \in I\}$. Use the Acyclic Covering Lemma to deduce that $H_q(L) \cong H_q(N)$ for all $q \geq 0$.

(e) Recall that $\mathbb{F}$ denotes the field of fractions of R. Let $\{W_j \mid j \in J\}$ denote the set of hyperplanes in $\mathbb{F}^n$. Let T_j be the subcomplex of the Tits building $\mathcal{T}_n(\mathbb{F})$ consisting of all simplices with W_j as a vertex. Show that $\{T_j \mid j \in J\}$ is an acyclic covering of L in the sense of the Acyclic Covering Lemma.

(f) Let $\tilde{N}$ be the nerve of the cover $\{T_j \mid j \in J\}$. Use the Acyclic Covering Lemma to deduce that $H_q(\mathcal{T}_n(\mathbb{F})) \cong H_q(\tilde{N})$ for all $q \geq 0$.

(g) Show that the mapping

$$\begin{aligned} \{\text{summands of } R^n\} &\longrightarrow \{\text{subspaces of } \mathbb{F}^n \cong R^n \otimes_R \mathbb{F}\} \\ H &\longmapsto H \otimes_R \mathbb{F} \end{aligned}$$

defines a simplicial isomorphism of N onto $\tilde{N}$.

(h) We have proved the following isomorphisms. Verify that they are $\mathrm{GL}_n(R)$-equivariant.

$$\begin{aligned} H_q(K, L) &\cong \widetilde{H}_{q-1}(L) \\ &\cong \widetilde{H}_{q-1}(N) \\ &\cong \widetilde{H}_{q-1}(\tilde{N}) \\ &\cong \widetilde{H}_{q-1}(\mathcal{T}_n(\mathbb{F})) \end{aligned}$$

(i) Verify that the $(n-2)$-skeleton of L coincides with the $(n-2)$-skeleton of K, so

$$C_q(K, L) = 0 \qquad \text{for } q \leq n-2.$$

(j) Using the Solomon–Tits result that $\mathcal{T}_n(\mathbb{F}) \simeq \bigvee S^{n-2}$, deduce that there is an exact sequence

$$\cdots \longrightarrow C_{q+n}(K, L) \longrightarrow C_{q+n-1}(K, L) \longrightarrow \cdots \longrightarrow C_{n-1}(K, L) \longrightarrow \mathrm{St}_n(R) \longrightarrow 0.$$

(k) Show that there are $\mathrm{GL}_n(R)$-equivariant isomorphisms of chain complexes

$$\begin{aligned} C_{q+n-1}(K) &\cong C(\mathscr{S}_q) \\ C_{q+n-1}(L) &\cong C(\mathscr{P}_q) \\ C_{q+n-1}(K, L) &\cong C_q(R^n) \end{aligned}$$

(l) Conclude the existence of the Sharbly resolution.

(m) Show that differentials in the Sharbly resolution are induced by the map

$$\mathscr{S}_q \longrightarrow \mathscr{S}_{q-1}$$
$$A \longmapsto \sum(-1)^i d_i(A)$$

where d_i is the map that deletes the i^{th} row of the matrix A.

(15) Prove that the Sharbly resolution is free.
Hint: First verify that $\mathrm{GL}_n(R)$ acts freely on the set of $(n+q)\times n$ matrices A satisfying
- each row of A is unimodular,
- some $(n\times n)$-submatrix of A has non-vanishing determinant.

(16) Give a geometric interpretation of the map $\phi : C_0(R^n) \to \mathrm{St}_n(R)$. Conclude that the Steinberg module is generated by apartment classes.

Let R be a Euclidean ring with a multiplicative Euclidean norm. Using the Sharbly resolution, Lee–Szczarba went on to prove that the $\mathrm{SL}_n(R)$–coinvariants of $C_0(R^n)$ vanish, which implies that

$$H_0(\mathrm{SL}_n(R); \mathrm{St}_n(R)) = 0.$$

By virtual Bieri–Eckmann duality, this then implies that the rational cohomology of $\mathrm{SL}_n(R)$ vanishes in its virtual cohomological dimension.

(17) **(Bonus).** Let R be a Euclidean ring with a multiplicative Euclidean norm. Prove directly that $\mathrm{SL}_n(R)$–coinvariants of $C_0(R^n)$ vanish. See Lee–Szczarba Theorem 4.1.

Ash–Rudolph

Let $\mathcal{T}_n(\mathbb{Q})$ be the Tits building on the rational vector space $\mathbb{Q}^n$. Let A_L denote the apartment associated to a frame $L = \{L_1, L_2, \ldots, L_n\}$ for $\mathbb{Q}^n$. Recall the following.

Definition (Integral apartment). Let $L = \{L_1, L_2, \ldots, L_n\}$. The frame L (and the apartment A_L) are called *integral* if

$$(L_1 \cap \mathbb{Z}^n) \oplus (L_2 \cap \mathbb{Z}^n) \oplus \cdots \oplus (L_n \cap \mathbb{Z}^n) = \mathbb{Z}^n.$$

(18) Consider $\mathbb{Q}^2$. Verify that the apartment corresponding to the frame

$$\left\{ \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix}, \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix} \right\}$$

is integral, but the apartment corresponding to the frame

$$\left\{ \mathbb{Q}\begin{bmatrix}2\\1\end{bmatrix}, \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix} \right\}$$

is not integral.

(19) Develop a determinant condition for verifying whether or not a frame is integral.

Let $L = \{L_1, L_2, \ldots, L_n\}$ be a frame, and let A_L be the associated apartment in the Tits building. Recall from Exercises (6) and (7) that A_L is an $(n-2)$-sphere, specifically, it is simplicially isomorphic to the barycentric subdivision of the boundary of an $(n-1)$-simplex.

(20) (a) A permutation $\sigma \in S_n$ acts on L by $L_i \mapsto L_{\sigma(i)}$. Show that σ induces a simplicial isomorphism $A_L \to A_L$. Show that this isomorphism is orientation-preserving if σ is even, and orientation-reversing if σ is odd.

(b) The apartment A_L represents a homology class $[A_L] \in H_{n-2}(\mathcal{T}_n(\mathbb{Q}))$. Explain why, to make the sign of $[A_L]$ well-defined, we must order the frame L up to sign. Moreover, the symmetric group $S_n \subseteq \mathrm{GL}_n(\mathbb{Z})$ acts on $[A_L]$ by sign.
(Some sources leave L unordered and $[A_L]$ only defined up to sign).

Ash–Ruldolph (1979) proved the following.

Theorem (Ash–Rudolph). The homology group $H_{n-2}(\mathcal{T}_n(\mathbb{Q}))$ is generated by integral apartment classes

$$\left\{ [A_L] \;\middle|\; \begin{array}{l} L \text{ an integral frame for } \mathbb{Q}^n, \\ \text{ordered up to sign} \end{array} \right\}.$$

(21) (a) Find an element of $\mathrm{SL}_n(\mathbb{Z})$ that interchanges the two lines

$$\left\{ \mathbb{Q}\begin{bmatrix}1\\0\end{bmatrix}, \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix} \right\}.$$

(b) Show that there is **no** matrix in $\mathrm{SL}_n(\mathbb{Z})$ that interchanges the two lines

$$\left\{ \mathbb{Q}\begin{bmatrix}2\\1\end{bmatrix}, \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix} \right\}.$$

(c) Explain why this non-existence result was an obstacle in our first lecture to computing $H_0(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q})$, and explain why it is resolved by Ash–Rudolph.

(22) **(Bonus).** Again consider $\mathbb{Q}^2$. Write the apartment class corresponding to the frame

$$\left\{ \mathbb{Q}\begin{bmatrix}2\\1\end{bmatrix}, \mathbb{Q}\begin{bmatrix}0\\1\end{bmatrix} \right\}$$

as a linear combination of integral apartment classes.

This theorem of Ash–Rudolph offers a simpler proof of the earlier theorem of Lee–Szczarba.

(23) (a) Let C_0 be the free abelian group on the symbols $[L]$ with $L = \{L_0, L_1, \ldots, L_n\}$ a frame for R^n, L ordered up to sign, subject to the relation

$$\sigma \cdot [L] = (-1)^{\mathrm{sign}(\sigma)}[L] \qquad \text{for } \sigma \in S_n.$$

Show that the $\mathrm{SL}_n(\mathbb{Z})$-coinvariants of C_0 vanish.

(b) Deduce that $H_0(\mathrm{SL}_n(\mathbb{Z}); \mathrm{St}_n(\mathbb{Q})) = 0$.
(c) Deduce that $H^{vcd}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0$.